Solutions to Explorations to Accompany

Polar Coordinates — Insights and Applications

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NCTM Annual Meeting, San Antonio, 4/7/17

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Availability of Materials:

Calculus: Concepts and Applications and *Precalculus with Trigonometry: Concepts and Applications,* (originally from Key Curriculum Press) along with the Explorations, solutions, and other ancillary material are available from Kendall Hunt.

To request a free 30-day on-line demo, do the following:

- Go to: http://flourishkh.com/main/request-demo
- Click on "Calculus" or "Precalculus", then on "Request Demo"
- Fill out the form and submit.
- When you get the demo, click on "Calculus" or "Precalculus."
- You can select the text itself or teaching resources.
- The Explorations and their solutions are under resources.

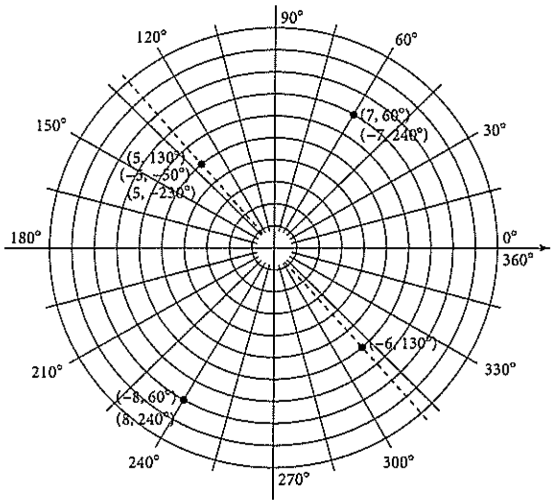
Algebra I: Equations, Expresions, and Applications, and *Algebra and Trigonometry: Functions and Applications* (originally from Addison-Wesley) and ancillary materials (other than the Explorations) are available from Pearson/Prentice Hall.

To order textbooks from the publisher:

- Go to: http://www.pearsonschool.com
- Click on the product search icon at the top.
- In the search box type "Foerster classics".
- Under Top Results, click on Algebra I and Algebra & Trigonometry — Classics Edition.
- Click on the pull-down menu titled Shop.
- Select either Algebra I or Algebra & Trigonometry.
- Add the materials you want to the cart.

For Explorations, contact "Paul Foerster"<foerster@idworld.net>

Problem Set 11-1



1. Point (7, 60°) on figure
2. Point (−8, 60°) on figure
3. (8, 240°) is at the same place as (−8, 60°). (−7, 240°) is at the same place as (7, 60°). Surprising!
4. Points (5, 130°) and (−6, 130°) on figure
5. Angle is 130° − 180° = −50°. Point is (−5, −50°), at the same place as (5, 130°).
6. Possible answer: Angle is 130° − 360° = −230°. Point is (5, −230°).
7. θ is the independent variable because you select θ first, then go out the appropriate displacement r. This disagrees with the custom of putting the independent variable first in an ordered pair.
8. Answers will vary.

Problem Set 11-2

Q1. $\sin\theta = \frac{b}{r}$

Q2. $\cos\theta = -\frac{a}{r}$

Q3. $\tan\theta = -\frac{b}{a}$

Q4. $\cot\theta = -\frac{a}{b}$

Q5. $\sec\theta = -\frac{r}{a}$

Q6. $\csc\theta = \frac{r}{b}$

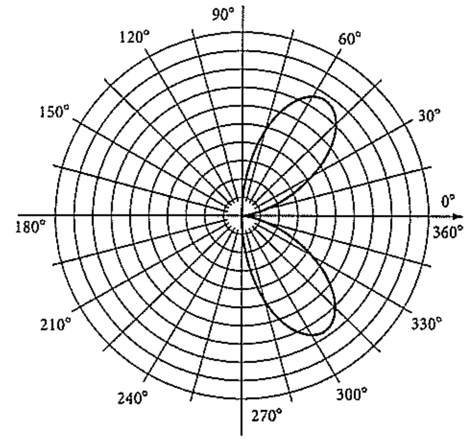
Q7. $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$

Q8. $\cos 150° = -\frac{\sqrt{3}}{2}$

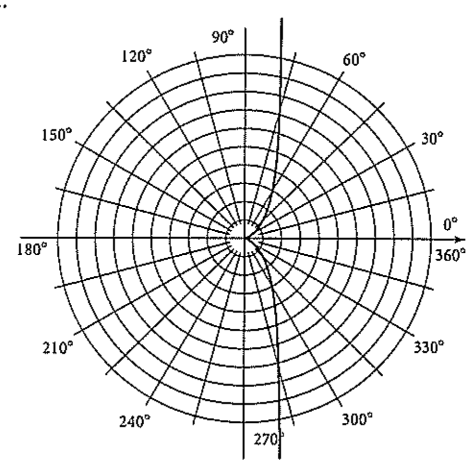
Q9. $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Q10. C

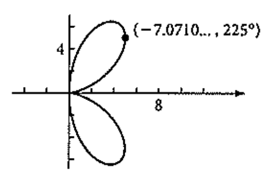
1.



2.

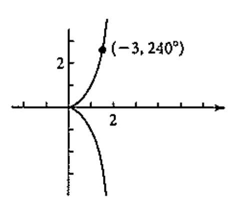


3.



The graph is being retraced between 180° and 360°. The figure has two (*bi-*) "leaves" (*folium*).

4.

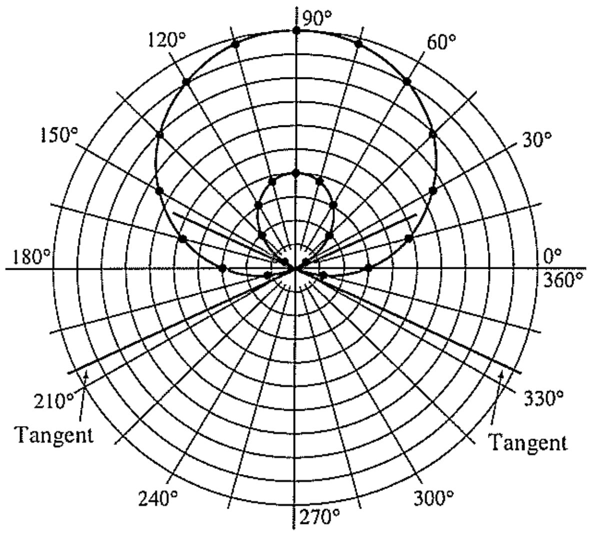


The root *cis-* means "on the same side" (opposite of *trans-*). Diocles (c. 240 B.C.E.–c. 180 B.C.E.) was a Greek mathematician who worked with cissoids in his attempt to duplicate a cube.

Name: ____________________ Group Members: ______________ ______________ ______________

Foerster Precalculus Solutions, Exploration 11-2 — Limaçon in Polar Coordinates Date: ____________

Objective: Plot polar curves on paper and on your grapher.



1. The following is a table of values of r and θ for a curve in polar coordinates. Plot the points on the polar coordinate paper above, and connect them with a smooth curve. • See graph

θ	r	θ	r
0°	3.0	195°	1.2
15°	4.8	210°	−0.5
30°	6.5	225°	−2.0
45°	7.9	240°	−3.1
60°	9.1	255°	−3.8
75°	9.8	270°	−4.0
90°	10.0	285°	−3.8
105°	9.8	300°	−3.1
120°	9.1	315°	−2.0
135°	7.9	330°	−0.5
150°	6.5	345°	1.2
165°	4.8	360°	3.0
180°	3.0		

2. Tell how you plot points for which r is negative.

• You rotate the number line so that the positive part is at the correct angle θ then go to the correct number in the negative direction to plot the point.

3. The equation of the curve in Problem 1 is

$r = 3 + 7 \sin \theta$

Do you agree that r-values from this equation, rounded to one decimal place, agree with the values in the table in Problem 1?

• The values agree to one decimal place.

4. Plot the graph in Problem 3 on your grapher. Use a θ-range of [0°, 360°] with a θ step of 5°. Use equal scales on the two axes. Did your grapher graph confirm the graph you plotted in Problem 1?

• Yes.

5. Check in your text to find the name of the geometrical figure in Problems 1 and 4.

• Limaçon (See also the title above.)

6. Set r equal to 0 in Problem 3 and solve the resulting equation for θ. Write the general solution.

• $3 + 7 \sin \theta = 0$

• $\sin \theta = -\frac{3}{7}$

• $\theta = \arcsin(-3/7)$
$= -25.3769...° + 360n°$ or $205.3769...° + 360n°$

7. Write the *two* values of θ in [0°, 360°] at which the graph goes through the pole.

• $\theta = 205.3796...°$ or $334.6230...°$

8. Draw a line on your graph in Problem 1 at each of the two values of θ in Problem 7. How are the rays related to the graph?

• Graph, above Problem 1.

• The lines are tangent to the graph at the points where the graph passes through the pole

9. What did you learn as a result of doing this Exploration that you did not know before?.

• Answers will vary.

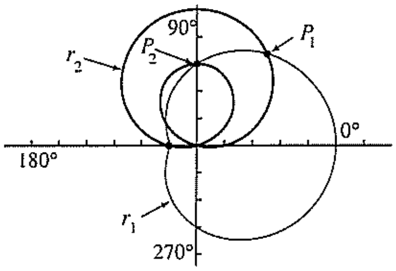
Name: ____________________ Group Members: _______________ ________________ ________________

Foerster Precalculus   Solutions, Exploration 11-3 rev — Intersections of Polar Curves   Date: ____________

Objective: Plot polar curves on your grapher and find places where polar curves intersect.

The figure shows

(1) the limaçon $r_1 = 3 + 2\cos\theta$ and
(2) the limaçon $r_2 = 1 + 4\sin\theta$.



1. Plot the two graphs on your grapher. Use degrees and simultaneous mode, and a fairly small θ-step so that the graphs plot relatively slowly. Pause the plotting when the graphs reach the intersection point P_1. Approximately what does θ equal at this point?

   • $\theta \approx 53°$

2. Resume the plotting, then pause it again at the θ-value corresponding to point P_2 on the r_1 limaçon. Where is the point on the r_2 limaçon for this value of θ? Explain why P_2 is not an intersection point of the two graphs.

   • At $\theta = 90°$, the r_1 graph and the r_2 graph are at different places.

3. Continue the graphing until a complete 360° has been plotted. Trace on the r_1 graph to the point P_2. Is there a simple relationship between the values of r_1 and r_2 for this value of θ?

   • There is no simple relationship between the 3 for the r_1 graph and the 5 for the r_2 graph at $\theta = 90°$

4. Trace on the r_2 graph the point P_2. Is there a simple relationship between the values of r_1 and r_2 for this value of θ? How is this value of θ related to the value of θ in Problem 3?

   • The r_2 graph is a P_2 for $\theta = 270°$.

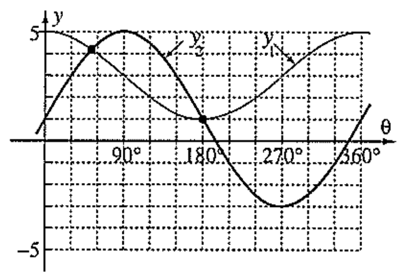
   • At 270°, $r_1 = 3$ and $r_2 = -3$, opposites.

   • 270° is 180° farther around from 90°.

5. With your grapher in function mode, plot the auxiliary Cartesian graphs

   $y_1 = 3 + 2\cos\theta$
   $y_2 = 1 + 4\sin\theta$

   Sketch the result.



6. Solve numerically to find the two values of θ in [0°, 360°] where the graphs in Problem 5 intersect. Show that these correspond to the two points the polar graphs intersect.

   • $\theta = 53.1301...°$ or 180°

   • These values agree with the graph above #1.

7. Solve algebraically for the points in Problem 6.

   • $3 + 2\cos\theta = 1 + 4\sin\theta$

   • $\cos\theta - 2\sin\theta = -1$

   • $\sqrt{5}\cos(\theta - \arctan(-2/1)) = -1$

   • $\cos(\theta + 63.4349...°) = -1/\sqrt{5}$

   • $\theta + 63.4349...° = \pm 116.560...° + 360n°$

   • $\theta = 53.1301...°$ or 180°

   • $(r, \theta) = (4.2, 53.1301...)$ or $(1, 180°)$

   • Note the *rational* r-value 4.2 for 53.1301...°

8. Explain why the coordinates of point P_2 do not show up on the auxiliary graph in Problem 5

   • The auxiliary graphs do not intersect at 90° or at 270°, the two possible values of θ for P_2.

9. Tell one new thing you learned as a result of doing this Exploration.

   • Answers will vary.

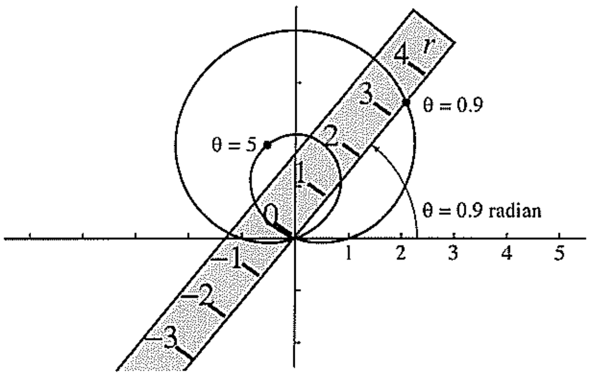
Name ____________________ Group Members: ______________________________

Solutions, Exploration 8-7a1: Polar Coordinate Calculus Date: __________

Objective: Find the area and arc length of a region bounded by graphs in polar coordinates, and intersections of polar curves.

A function in polar coordinates has a graph that is generated by a number line, the ***r*-axis**, as it rotates around the origin, the **pole**. The figure shows the graph of the **limaçon**

$r = 1 + 3\sin\theta$



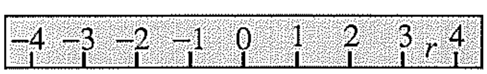
1. Plot the entire graph (one revolution) on your grapher. Use the window show, then press ZOOM SQUARE to make sure the scales are equal in both directions. Does your grapher graph agree with the given graph?

   • Grapher graph agrees.

2. On the FORMAT menu, select POLAR GRAPH COORDINATES. Then trace to $\theta = 0.9$ radian. Does the result agree with the point shown on the given graph? If so, record the value of r.

   $r =$ • 3.3499...

3. Cut out a "ruler" from an index card and mark it to form an r–number line as shown here.



   Use the scales on the figure above. Mark both positive and negative values of r.

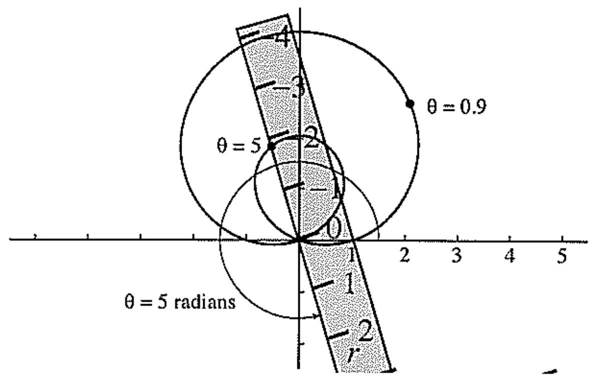
4. Place this r-number line along the polar axis in the figure, then rotate it about the pole to align it with the point on the graph where $\theta = 0.9$ radian. Draw the r-axis in this position, including the tick marks and numbers (positive and negative). Does the value of r agree with Problem 2?

   • Graph, above Problem 1

   • r-value of about 3.4 agrees with Problem 2.

5. Rotate the r-number line to the $\theta = 5$ radians. Note that when the positive part of the number line is at 5 radians, the negative part will pass thought the point on the figure marked $\theta = 5$. On this copy of the figure, draw the r-axis in this position, including the tick marks and numbers. Write the approximate value of r.

   $r \approx$ • −1.9

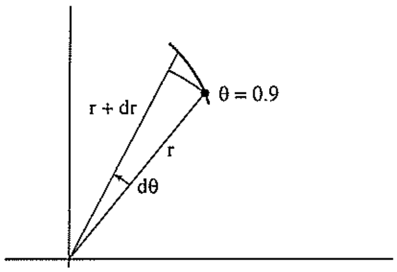


6. On your grapher, trace to $\theta = 5$. Does the value of r on the grapher agree with the value of r you wrote in Problem 5? If not, fix both your drawing and your value of r in Problem 5.

   $r =$ • −1.8767...

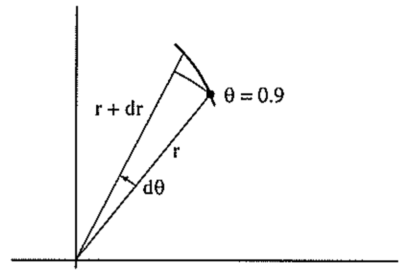
   Agree? • Yes.

(Other side)



7. ***Area Problem:*** The figure above zooms in on the graph around $\theta = 0.9$. The area of the region swept out by the r-axis as θ increases from 0.9 to $0.9 + d\theta$ is approximately equal to the area of the sector of a circle of radius r and angle $d\theta$. Write an equation for dA, the differential of area, in terms of r and $d\theta$. Then integrate to find the total area of the region in Quadrant I that is bounded by the outer loop of the limaçon.

- Area of circle is πr^2.
- The sector is $\dfrac{d\theta}{2\pi}$ of the circle.
- So the area of the sector is $dA = \pi r^2 \cdot \dfrac{d\theta}{2\pi}$
- $dA = \frac{1}{2} r^2\, d\theta$
- $dA = \frac{1}{2}(1 + 3\sin\theta)^2\, d\theta$
- $A = \frac{1}{2}\int_0^{\pi/2} (1 + 3\sin\theta)^2\, d\theta$
- $\underline{A = 7.3196...}$



8. ***Arc Length Problem:*** The length of the polar curve generated as θ increases from 0.9 to $0.9 + d\theta$ is approximately equal to the "hypotenuse," dL, of the "right triangle" whose "legs" are the arc of the circle of radius r and the change in the radius, dr. Write an equation for dL, the differential of arc length. Then integrate to find the total length of the curve (inner and outer loops).

- Circumference of circle is $2\pi r$.
- The arc is $\dfrac{d\theta}{2\pi}$ of the circle.
- So the length of the arc is $2\pi r \cdot \dfrac{d\theta}{2\pi} = r\, d\theta$
- $dL = \sqrt{dr^2 + (r\, d\theta)^2}$
- $dL = \sqrt{(3\cos\theta\, d\theta)^2 + ((1 + 3\sin\theta)\, d\theta)^2}$
- $dL = \sqrt{(3\cos\theta)^2 + (1 + 3\sin\theta)^2}\; d\theta$
- $L = \int_0^{\pi/2} \sqrt{(3\cos\theta)^2 + (1 + 3\sin\theta)^2}\; d\theta$
- $L = \underline{19.3768...}$

9. What did you learn as a result of working this exploration that you did not know before?

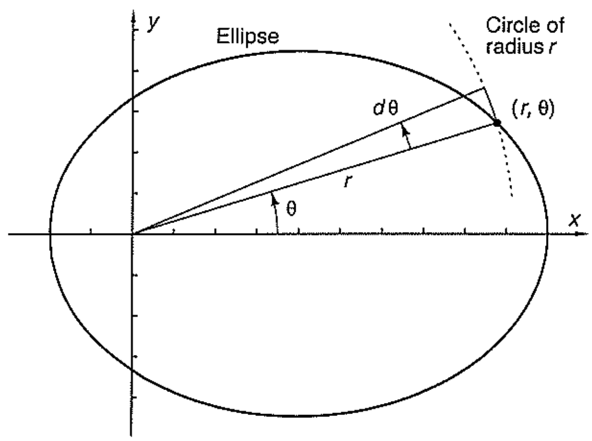
- Answers will vary, but should include the meaning of negative values of r, and the methods for calculating length of a polar curve and area of a region bounded by the curve, Name: ____________________ Group Members: ____

Foerster Calculus Solutions, Exploration 8-7a: Area of an Ellipse in Polar Coordinates Date: ______________

Objective: Find the area of an ellipse from its polar equation, then compare with the area found by familiar geometry.

The figure shows the **ellipse** in polar coordinates

$$r = \frac{10}{3 - 2\cos\theta}.$$



1. Set your grapher to POLAR mode and plot the graph. Use equal scales on both x- and y-axes. Does your graph agree with the figure above?

   • Graph agrees.

2. The sample point (r, θ) shown in the figure is at $\theta = 0.3$ radian. Calculate r, x, and y for this point. Show that all three agree with the graph.

   • $r = 10/(3 - 2\cos\theta) = 9.1799...$

   • $x = 9.1799... \cos 0.3 = 8.7699...$

   • $y = 9.1799... \sin 0.3 = 2.7128...$

3. The area of a wedge-shaped piece of the elliptical region bounded by the graph is approximately equal to the area of the sector of a circle. Calculate the area of the sector shown in the figure above, if $\theta = 0.3$ radian and $d\theta = 0.1$ radian.

   • Area $= \pi(9.1799...)^2(0.1/(2\pi))$

   $= 4.2136...$ square units

4. Show that in general, the area dA of the sector is

   $dA = \frac{1}{2} r^2\, d\theta$.

   • $dA = \pi r^2\, (d\theta/(2\pi)) = \frac{1}{2} r^2\, d\theta$, Q.E.D.

5. The exact area of the elliptical region is the *limit* of the *sum* of the sectors' areas. That is, the area equals the definite integral of dA. Write an integral representing the exact area. Evaluate the integral numerically.

   • $dA = \frac{1}{2}(10/(3 - 2\cos\theta))^{2}\, d\theta$

   • $A = \int_0^{2\pi} dA$

   $= 84.2977...$

6. The x-radius, a, of the ellipse shown is 6 units. Measure or calculate the y-radius, b. Then confirm that the answer you got by integration in Problem 5 agrees with the answer you get using the ellipse area formula $A = \pi ab$.

   • y -radius ≈ 4.5 (exactly sq. root of 20)

   • $A \approx \pi(6)(4.5) = 84.82...$

   • Answer is close to value in Problem 5.

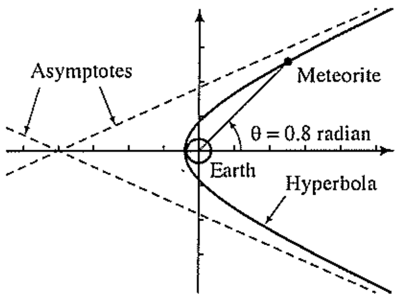
7. What did you learn as a result of doing this Exploration that you did not know before?

   • Answers will vary.

Name ____________________ Group Members: ______________________________

Solutions, Exploration 8-7c: Meteorite Polar Coordinates Problem Date: ________

Objective: Find the area and arc length of a region bounded by graphs in polar coordinates.



When a meteorite approaches the Earth it is drawn into a more and more curved path by gravity as it gets closer and closer. If the velocity is high enough, the meteorite follows a hyperbolic path with the Earth's center at one focus as shown in the figure. (When the meteorite is far away, its path is close to one of the two asymptotes of the hyperbola.) Assume that a meteorite is on a path whose polar equation is

$$r = 84(10 - 11\cos\theta)^{-1}$$

where distances are in thousands of miles.

1. Plot the path on your grapher. Use a window with [–55, 55] for x and equal scales on the two axes. Does your graph agree with the figure?
   - Graph agrees with figure.
2. How far is the meteorite from the Earth's center at the point shown in the figure where $\theta = 0.8$ rad.?
   - $r(0.8) = 84(10 - 11\cos 0.8)^{-1} = 35.9554...$ ≈ 35.955 thousand miles
3. Find the area swept out by the line segment from the Earth's center to the meteorite from the time $\theta = 0.8$ to the time $\theta = 5.5$ Show your work.
   - $A = \int_{0.8}^{5.5} 0.5(84(10 - 11\cos\theta)^{-1})^2\,d\theta$
     $= 308.9953...$
     ≈ 308.995 million square miles
4. How far does the meteorite travel along its curved path from the time $\theta = 0.8$ to the time $\theta = 5.5$? Show your work.
   - $dL = \sqrt{dr^2 + (r\,d\theta)^2}$
   - $dr = -84(10 - 11\cos\theta)^{-2}(11\sin\theta)\,d\theta$
   - $L = \int_{\theta=0.8}^{5.5} dL = 82.4852...$
     ≈ 82.485 thousand miles
5. Perform a quick check to show that the distance you calculated in Problem 4 is reasonable.
   - Check: $L > r(0.8) + r(5.5)$
     $L > 35.9554... + 38.1015... = 74.0570...$,
     $82.4852... > 74.0570...$, which checks.
6. What did you learn as a result of doing this Exploration that you did not know before?
   - Answers will vary.