Explorations to Accompany

Polar Coordinates — Insights and Applications

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(For solutions and additional copies, email "Paul Foerster"foerster@idworld.net)

Availability of Materials:

Calculus: Concepts and Applications and *Precalculus with Trigonometry: Concepts and Applications,* (originally from Key Curriculum Press) along with the Explorations, solutions, and other ancillary material are available from Kendall Hunt.
To request a free 30-day on-line demo, do the following:

- Go to: http://flourishkh.com/main/request-demo
- Click on "Calculus" or "Precalculus", then on "Request Demo"
- Fill out the form and submit.
- When you get the demo, click on "Calculus" or "Precalculus."
- You can select the text itself or teaching resources.
- The Explorations and their solutions are under resources.

Algebra I: Equations, Expresions, and Applications, and *Algebra and Trigonometry: Functions and Applications* (originally from Addison-Wesley) and ancillary materials (other than the Explorations) are available from Pearson/Prentice Hall.
To order textbooks from the publisher:

- Go to: http://www.pearsonschool.com
- Click on the product search icon at the top.
- In the search box type "Foerster classics".
- Under Top Results, click on Algebra I and Algebra & Trigonometry — Classics Edition.
- Click on the pull-down menu titled Shop.
- Select either Algebra I or Algebra & Trigonometry.
- Add the materials you want to the cart.

For Explorations, contact "Paul Foerster"<foerster@idworld.net>

Exploratory Problem Set 11-1

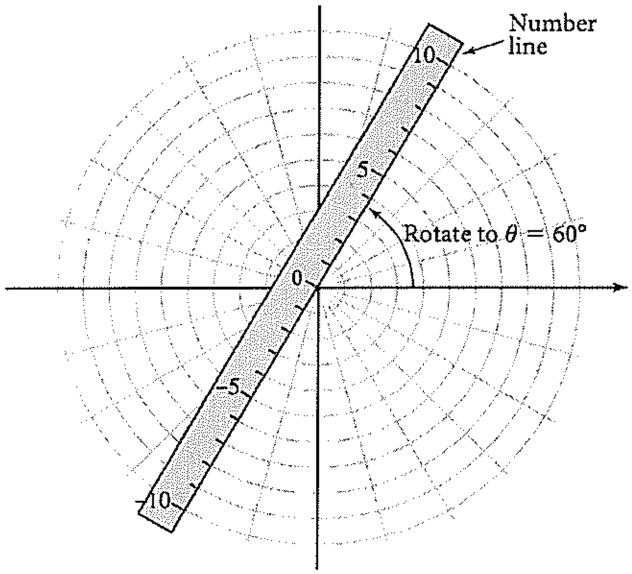


Figure 11-1a

1. Obtain a piece of polar coordinate graph paper. As shown in Figure 11-1a, make a "number-line" ruler out of paper or an index card and mark it off with the same scale as the polar coordinate paper. Put the origin (zero) of the number line at the **pole** (the origin on the polar coordinate paper). Then rotate the number line counterclockwise to $\theta = 60°$ in standard position. On the polar coordinate paper, draw a ray at 60° and mark the angle. On this ray, mark the point $(r, \theta) = (7, 60°)$ by going 7 units in the positive direction along the number line. Write $(7, 60°)$ at this point.

2. With the number-line ruler at the same place, go to -8 units. Extend the ray in Problem 1 backward using a dashed line, and mark the point $(r, \theta) = (-8, 60°)$.

3. Rotate the number-line ruler to $\theta = 240°$. Plot the points $(r, \theta) = (8, 240°)$ and $(-7, 240°)$. What do you notice about these two points?

4. Rotate the number-line ruler to $\theta = 130°$. Mark the points $(r, \theta) = (5, 130°)$ and $(r, \theta) = (-6, 130°)$ and their coordinates on the polar coordinate paper. Draw a solid ray representing the terminal side of the 130° angle and a dashed ray extending in the opposite direction for the negative value of r.

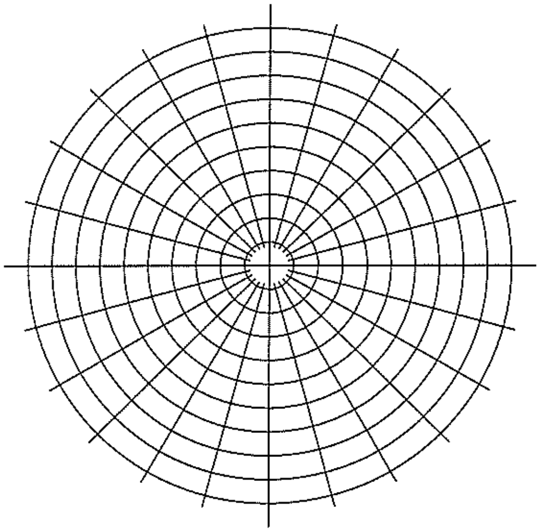
5. Rotate the number-line ruler clockwise far enough for its negative side to pass through the point $(r, \theta) = (5, 130°)$ from Problem 4. Then write another ordered pair for this point using the appropriate negative angle.

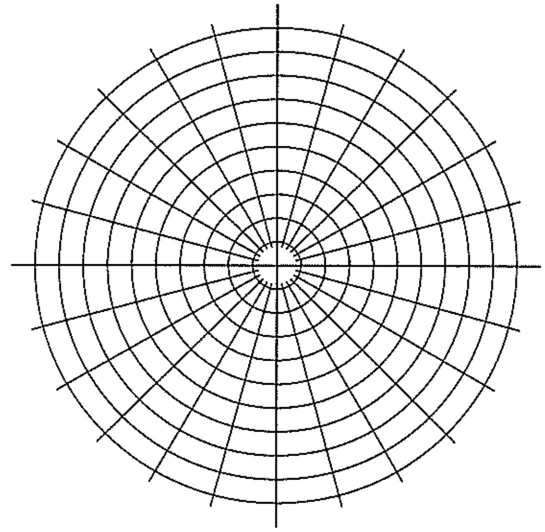
6. Write a third ordered pair (r, θ) for the point in Problem 5 using a negative angle and a positive value of r.

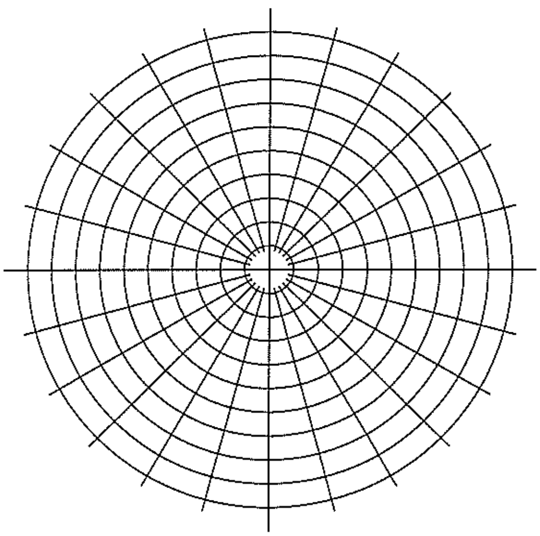
7. Based on the way you plotted the points in this exploration, which is the independent variable, r or θ? Does this agree or disagree with the custom of putting the independent variable first in an ordered pair?

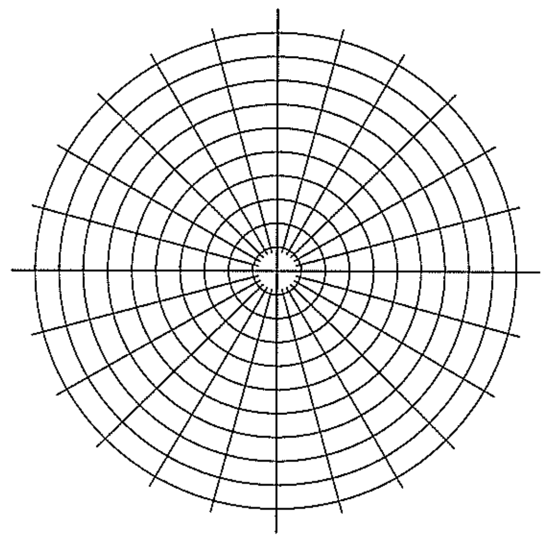
8. What did you learn as a result of doing this problem set that you did not know before?

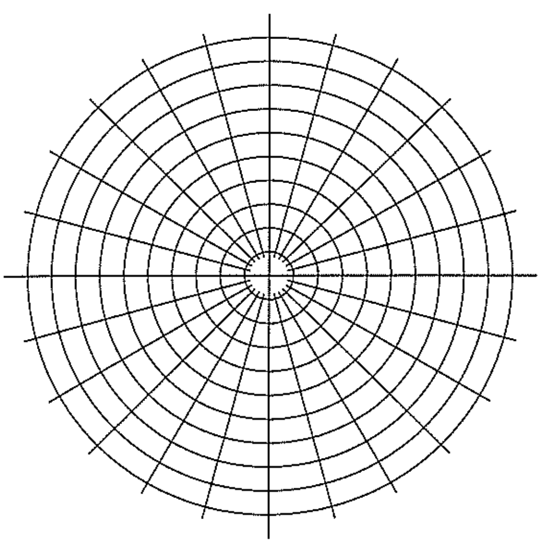
Name: ________________________ Polar Coordinate Paper Date: _______________

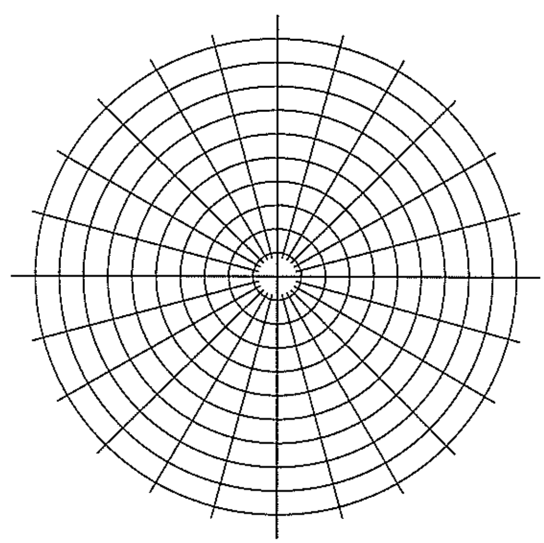








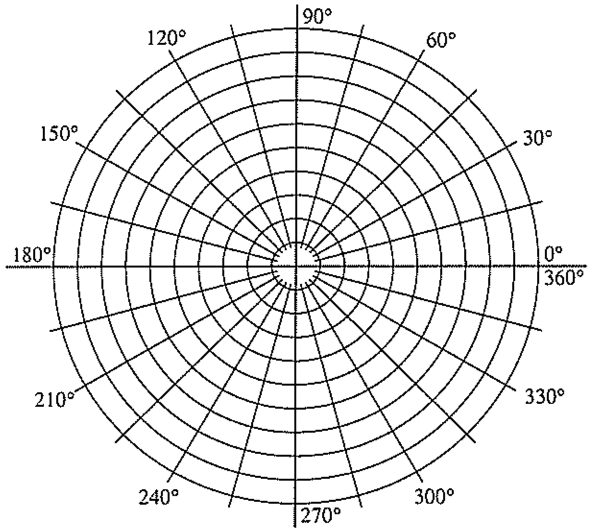




Name: ____________________ Group Members: _______________ _________________ _______________

Foerster Precalculus   Exploration 11-2 — Limaçon in Polar Coordinates   Date: ____________

Objective: Plot polar curves on paper and on your grapher.



1. The following is a table of values of r and θ for a curve in polar coordinates. Plot the points on the polar coordinate paper above, and connect them with a smooth curve.

θ	r	θ	r
0°	3.0	195°	1.2
15°	4.8	210°	–0.5
30°	6.5	225°	–2.0
45°	7.9	240°	–3.1
60°	9.1	255°	–3.8
75°	9.8	270°	–4.0
90°	10.0	285°	–3.8
105°	9.8	300°	–3.1
120°	9.1	315°	–2.0
135°	7.9	330°	–0.5
150°	6.5	345°	1.2
165°	4.8	360°	3.0
180°	3.0		

2. Tell how you plot points for which r is negative.

3. The equation of the curve in Problem 1 is

   $r = 3 + 7 \sin \theta$

   Do you agree that r-values from this equation, rounded to one decimal place, agree with the values in the table in Problem 1?

4. Plot the graph in Problem 3 on your grapher. Use a θ-range of [0°, 360°] with a θ step of 5°. Use equal scales on the two axes. Did your grapher graph confirm the graph you plotted in Problem 1?

   ____

5. Check in your text to find the name of the geometrical figure in Problems 1 and 4.

   ____

6. Set r equal to 0 in Problem 3 and solve the resulting equation for θ. Write the general solution.

7. Write the *two* valuesod θ in [0°, 360°] at which the graph goes through the pole.

8. Draw a ray on your graph in Problem 1 at each of the two values of θ in Problem 7. How are the rays related to the graph?

9. What did you learn as a result of doing this Exploration that you did not know before?.

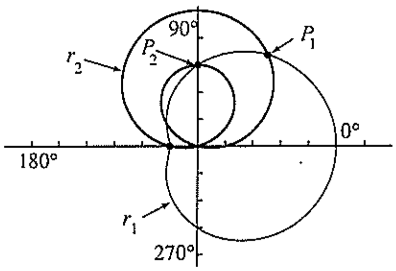
Name: ____________________ Group Members: _______________ __________________ ________________

Foerster Precalculus Exploration 11-3 rev — Intersections of Polar Curves Date: ____________

Objective: Plot polar curves on your grapher and find places where polar curves intersect.

The figure shows

(1) the limaçon $r_1 = 3 + 2\cos\theta$ and
(2) the limaçon $r_2 = 1 + 4\sin\theta$.



1. Plot the two graphs on your grapher. Use degrees and simultaneous mode, and a fairly small θ-step so that the graphs plot relatively slowly. Pause the plotting when the graphs reach the intersection point P_1. Approximately what does θ equal at this point?

2. Resume the plotting, then pause it again at the θ-value corresponding to point P_2 on the r_1 limaçon. Where is the point on the r_2 limaçon for this value of θ? Explain why P_2 is not an intersection point of the two graphs.

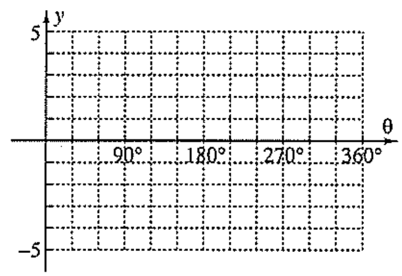
3. Continue the graphing until a complete 360° has been plotted. Trace on the r_1 graph to the point P_2. Is there a simple relationship between the values of r_1 and r_2 for this value of θ?

4. Trace on the r_2 graph the point P_2. Is there a simple relationship between the values of r_1 and r_2 for this value of θ? How is this value of θ related to the value of θ in Problem 3?

5. With your grapher in function mode, plot the auxiliary Cartesian graphs

$y_1 = 3 + 2\cos\theta$
$y_2 = 1 + 4\sin\theta$

Sketch the result.



6. Solve numerically to find the two values of θ in [0°, 360°] where the graphs in Problem 5 intersect. Show that these correspond to the two points the polar graphs intersect.

7. Solve algebraically for the points in Problem 6.

8. Explain why the coordinates of point P_2 do not show up on the auxiliary graph in Problem 5

9. Tell one new thing you learned as a result of doing this Exploration.

Name ______________________ Group Members: ____

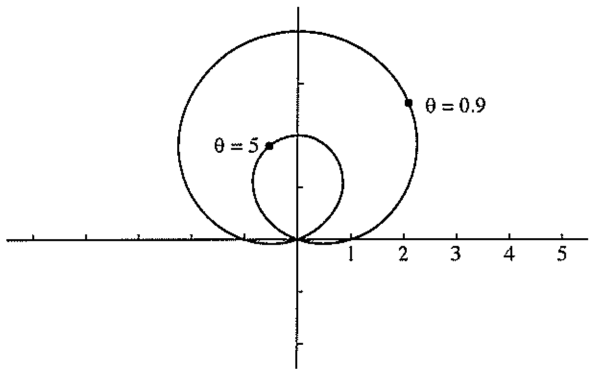
Exploration 8-7a1: Polar Coordinate Calculus

Date: __________

Objective: Find the area and arc length of a region bounded by graphs in polar coordinates, and intersections of polar curves.

A function in polar coordinates has a graph that is generated by a number line, the ***r*-axis**, as it rotates around the origin, the **pole**. The figure shows the graph of the **limaçon**

$r = 1 + 3 \sin \theta$

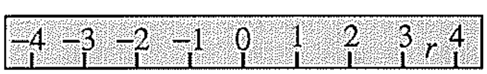


1. Plot the entire graph (one revolution) on your grapher. Use the window show, then press ZOOM SQUARE to make sure the scales are equal in both directions. Does your grapher graph agree with the given graph?

2. On the FORMAT menu, select POLAR GRAPH COORDINATES. Then trace to $\theta = 0.9$ radian. Does the result agree with the point shown on the given graph? If so, record the value of r.

   $r =$ ____________

3. Cut out a "ruler" from an index card and mark it to form an r–number line as shown here.

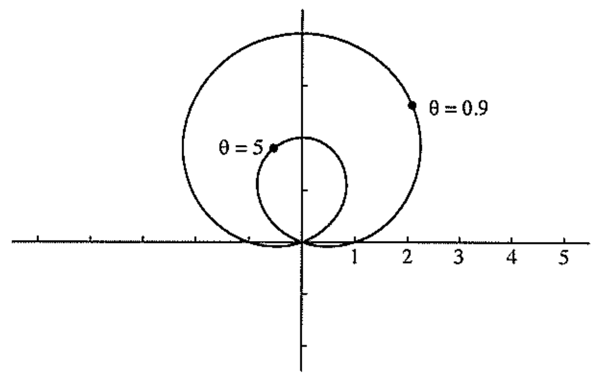


   Use the scales on the figure above. Mark both positive and negative values of r.

4. Place this r-number line along the polar axis in the figure, then rotate it about the pole to align it with the point on the graph where $\theta = 0.9$ radian. Draw the r-axis in this position, including the tick marks and numbers (positive and negative). Does the value of r agree with Problem 2?

5. Rotate the r-number line to the $\theta = 5$ radians. Note that when the positive part of the number line is at 5 radians, the negative part will pass thought the point on the figure marked $\theta = 5$. On this copy of the figure, draw the r-axis in this position, including the tick marks and numbers. Write the approximate value of r.

   $r \approx$ ____________

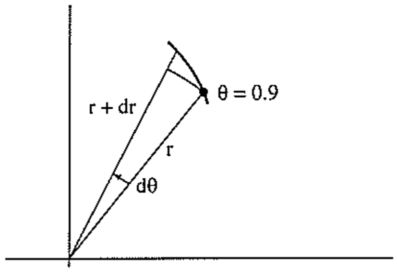


6. On your grapher, trace to $\theta = 5$. Does the value of r on the grapher agree with the value of r you wrote in Problem 5? If not, fix both your drawing and your value of r in Problem 5.

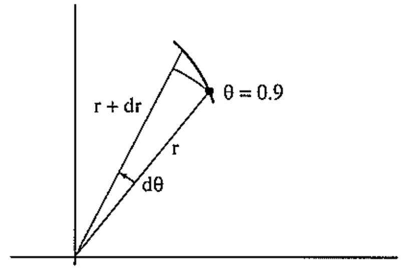
   $r =$ ____________

   Agree? _________

(Other side)



7. ***Area Problem:*** The figure above zooms in on the graph around $\theta = 0.9$. The area of the region swept out by the r-axis as θ increases from 0.9 to $0.9 + d\theta$ is approximately equal to the area of the sector of a circle of radius r and angle $d\theta$. Write an equation for dA, the differential of area, in terms of r and $d\theta$. Then integrate to find the total area of the region in Quadrant I that is bounded by the outer loop of the limaçon.



8. ***Arc Length Problem:*** The length of the polar curve generated as θ increases from 0.9 to $0.9 + d\theta$ is approximately equal to the "hypotenuse," dL, of the "right triangle" whose "legs" are the arc of the circle of radius r and the change in the radius, dr. Write an equation for dL, the differential of arc length. Then integrate to find the total length of the curve (inner and outer loops).

9. What did you learn as a result of working this exploration that you did not know before?

Name: ______________ Group Members: ______________________________

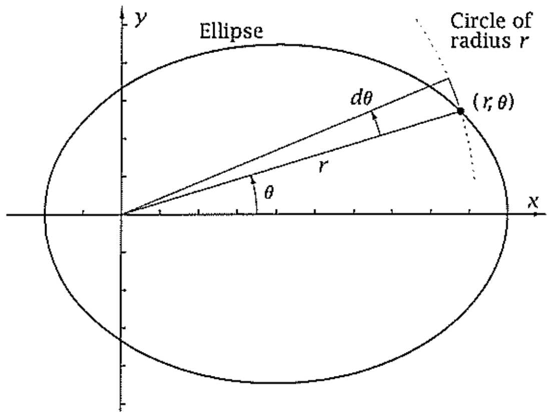
Exploration 8-7a: Area of an Ellipse in Polar Coordinates

Date: __________

Objective: Find the area of an ellipse from its polar equation, then compare with the area found by familiar geometry.

The figure shows the **ellipse** in polar coordinates

$$r = \frac{10}{3 - 2\cos\theta}$$



1. Set your grapher to POLAR mode and plot the graph. Use equal scales on both x- and y-axes. Does your graph agree with the figure above?

2. The sample point (r, θ) shown in the figure is at $\theta = 0.3$ radian. Calculate r, x, and y for this point. Show that all three agree with the graph.

3. The area of a wedge-shaped piece of the elliptical region bounded by the graph is approximately equal to the area of the sector of a circle. Calculate the area of the sector shown in the figure above, if $\theta = 0.3$ radian and $d\theta = 0.1$ radian.

4. Show that in general, the area dA of the sector is

$$dA = \frac{1}{2}r^2\,d\theta$$

5. The exact area of the elliptical region is the *limit* of the *sum* of the sectors' areas. That is, the area equals the definite integral of dA. Write an integral representing the exact area. Evaluate the integral numerically.

6. The x-radius, a, of the ellipse shown is 6 units. Measure or calculate the y-radius, b. Then confirm that the answer you got by integration in Problem 5 agrees with the answer you get using the ellipse area formula $A = \pi ab$.

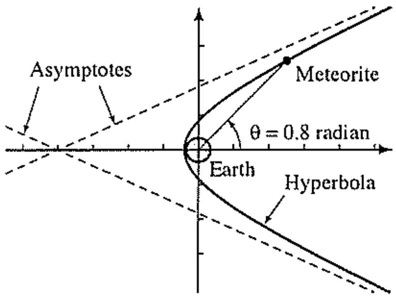
7. What did you learn as a result of doing this Exploration that you did not know before?

Name ________________________ Group Members: ____

Exploration 8-7c: Meteorite Polar Coordinates Problem

Date: ____________

Objective: Find the area and arc length of a region bounded by graphs in polar coordinates.



When a meteorite approaches the Earth it is drawn into a more and more curved path by gravity as it gets closer and closer. If the velocity is high enough, the meteorite follows a hyperbolic path with the Earth's center at one focus as shown in the figure. (When the meteorite is far away, its path is close to one of the two asymptotes of the hyperbola.) Assume that a meteorite is on a path whose polar equation is

$$r = 84(10 - 11\cos\theta)^{-1}$$

where distances are in thousands of miles.

1. Plot the path on your grapher. Use a window with $[-55, 55]$ for x and equal scales on the two axes. Does your graph agree with the figure?

2. How far is the meteorite from the Earth's center at the point shown in the figure where $\theta = 0.8$ rad.?

3. Find the area swept out by the line segment from the Earth's center to the meteorite from the time $\theta = 0.8$ to the time $\theta = 5.5$ Show your work.

4. How far does the meteorite travel along its curved path from the time $\theta = 0.8$ to the time $\theta = 5.5$? Show your work.

5. Perform a quick check to show that the distance you calculated in Problem 4 is reasonable.

6. What did you learn as a result of doing this Exploration that you did not know before?