Notes and Definitions

The symbolic definitions are included to help students learn how to read and interpret notation.

A ***sequence*** is a list of numbers in order.

- The *n* in the subscript of A_n may be the number of the term in the sequence or somehow related to the term. The terms of some sequence are given by a formula involving *n*.
- A more precise definition is that a sequence is a function whose domain is integers. Therefore, A_n could be written *A*(*n*).
- The first value of *n* is usually 0 or 1, but any integer may be used.

A ***non-decreasing sequence*** is one whose terms increase or remain the same from term to term.

Each member of the sequence is less than or equal to the next member.

- A sequence is non-decreasing if, and only if, for all *n*, $L_n \leq L_{n+1}$

A ***non-increasing sequence*** is one whose terms decrease or remain the same from term to term. Each member of the sequence greater than or equal to the next member.

- A sequence is non-increasing if, and only if, for all *n*, $G_n \geq G_{n+1}$

A sequence is ***bounded above*** if there exist a number greater than or equal to all the terms of the sequence.

- The smallest upper bound of a sequence is called its **least upper bound (*l.u.b.*).** The l.u.b. may or may not be a term of the sequence.
- A non-decreasing sequence that has an upper bound, approaches as a limit its l.u.b.

A sequence is ***bounded below*** if there exists a number less than or equal to all the terms of the sequence.

- The largest lower bound is called the **greatest lower bound (*g.l.b*)**. The g.l.b. may or may not be a term of the sequence.
- A non-increasing sequence that has a lower bound, approaches as a limit its g.l.b.

The ***limit of a sequence*** is the number that numbers in a sequence approach (or possibly equal) as *n* increases without bound.

- The notation is $\lim_{n \to \infty} A_n$. This expression is read, "The limit of *An* as n approaches infinity."
- The expression $n \to \infty$ is read "as *n* approaches infinity." It means that *n* gets larger without bound or gets larger than all (any, every) positive numbers. Infinity is not a number; you may not do arithmetic with it.
- Not all sequences have limits.

Additional information for teachers

The table values for the lesson are:

n = decimal places	L_n	G_n
0	1	2
1	1.4	1.5
2	1.41	1.42
3	1.414	1.415
4	1.4142	1.4143
5	1.41421	1.41422
6	1.414213	1.414214
7	1.4142135	1.4142136
8	1.41421356	1.41423157
9	1.414213562	1.414213563
10	1.4142135623	1.4142135624
11	1.41421356237	1.41421356238
12	1.414213562373	1.414213562374
13	1.4142135623730	1.4142135623731
14	1.41421356237309	1.41421356237310
15	1.414213562373095	1.414213562373096
16	1.4142135623730950	1.4142135623730951
17	1.41421356237309504	1.41421356237309505

WolframAlpha uses this algorithm to compute square roots. Notice that this, too, produces a sequence of numbers.

A sequence of approximations a/b to $\sqrt{n}$ can be derived by factoring

$$a^2 - n b^2 = \pm 1$$

(where -1 is possible only if -1 is a quadratic residue of n). Then

$$(a + b\sqrt{n})(a - b\sqrt{n}) = \pm 1$$

$$(a + b\sqrt{n})^k (a - b\sqrt{n})^k = (\pm 1)^k = \pm 1,$$

and

$$\begin{aligned} (1+\sqrt{n})^1 &= 1+\sqrt{n} \\ (1+\sqrt{n})^2 &= (1+n)+2\sqrt{n} \\ (1+\sqrt{n})(a+b\sqrt{n}) &= (a+b\,n)+\sqrt{n}\,(a+b). \end{aligned}$$

Therefore, a and b are given by the recurrence relations

$$\begin{aligned} a_i &= a_{i-1} + b_{i-1}\,n \\ b_i &= a_{i-1} + b_{i-1} \end{aligned}$$

with $a_1 = b_1 = 1$. The error obtained using this method is

$$\left|\frac{a}{b} - \sqrt{n}\right| = \frac{1}{b\left(a+b\sqrt{n}\right)} < \frac{1}{2b^2}.$$