Name ______________________________

An Exploration of an Interesting Function

Consider the differential equation $\frac{dy}{dx} = x(y-3)^2$.

1. Solve the differential equation by separating the variables and verify that the general solution of the differential equation is $y(x) = 3 - \frac{2}{x^2 + C}$ where *C* is a constant.

2. Check the solution by substituting the solution and its first derivative into the differential equation.

3. Using a graphing utility such as Desmos (https://www.desmos.com/), GeoGebra (https://www.geogebra.org/graphing), or your graphing calculator, examine the various possible graphs of the solution. There are two different forms depending on the values of $C \neq 0$. Sketch a graph of each. Let Type I be the solution with no vertical asymptote and Type II be the solutions with vertical asymptotes. Note the horizontal asymptote of each.

4. For the graphs with vertical asymptotes, what must be true about *C*? Find the equations of the vertical asymptotes in terms of *C*.

5. Find the particular solutions that have the following initial conditions.

   a. $f(0) = -2$

   b. $f(-1) = 4$

   c. $f(3) = 1$

6. The domain of a solution of a differential equation is (1) a continuous open interval, (2) that contains the initial condition, and (3) satisfies the differential equation. Find the domain of each solution found in part 5 above. Hint: Work from the graphs.

7. Extreme Value

   a. Show that all solutions with $C \neq 0$, have a minimum value. Justify your answer.

   b. Find the coordinates of the minimum point in terms of *C* when $C \neq 0$.

   c. Discuss the why there is no minimum when *C* = 0. What happen in this instance?

   d. Find the coordinates of the minimum point for the three solutions found in 5 above.

8. Concavity

   a. Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*.

   b. For each of the initial conditions in 5 above, find the value of $\frac{d^2y}{dx^2}$, and determine the concavity near the initial condition point? Does this agree with the graph?

9. Consider the special case of $y(x)=3$ a constant function.

   a. Show that this is a solution by substituting it and its first derivative into the differential equation.

   b. Determine what value of *C*, if any, produces this solution. (Hint: use any point on $y(x)=3$ as the initial condition.)

   c. Find $\lim_{C\to\infty}\left(3-\frac{2}{x^2+C}\right)=$ _____ and $\lim_{C\to-\infty}\left(3-\frac{2}{x^2+C}\right)=$ _____

   d. This function is not covered by, and therefore is not a contradiction of, the solution found in question 1 above. Why?