An Exploration of an Interesting Function – Answers

1. Solve the differential equation by separating the variable and verify that the general solution of the differential equation is $y(x)=3-\dfrac{2}{x^2+C}$ where *C* is a constant.

$$\int (y-3)^{-2}\,dy=\int x\,dx$$

$$-(y-3)^{-1}=\tfrac{1}{2}x^2+C_1$$

$$y-3=\frac{-2}{x^2+C},\ \text{where } C=\frac{C_1}{2}$$

$$y=3-\frac{2}{x^2+C}$$

2. Check the solution by substituting the solution and its first derivative into the differential equation.

$$\frac{dy}{dx}=\left(x^2+C\right)^{-2}(2x)$$

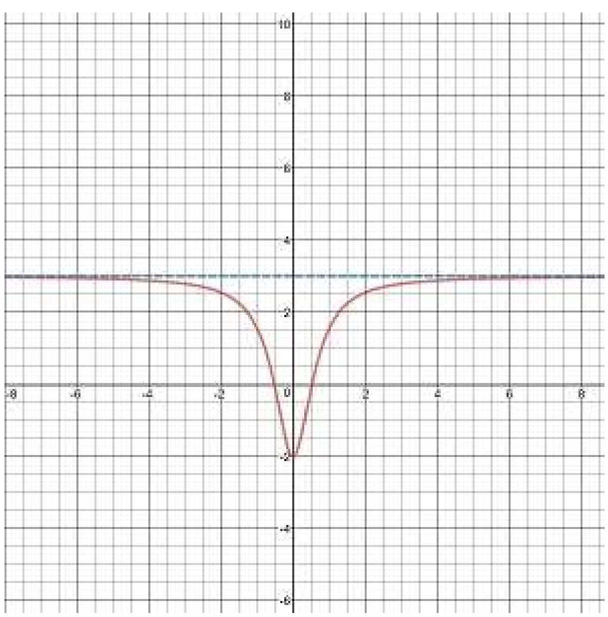
Substituting into the differential equation gives

$$\left(x^2+C\right)^{-2}(2x)=x\left(\left(3-\frac{2}{x^2+C}\right)-3\right)^2$$

Which checks

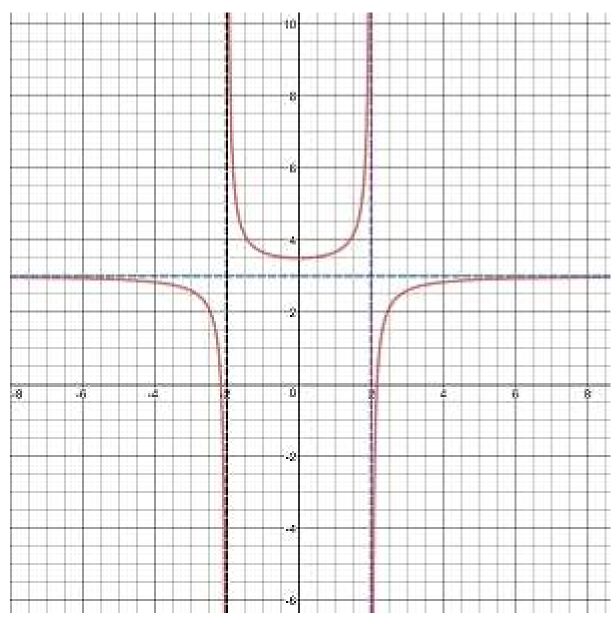
3. Type I: $C > 0$

C = 0.4; asymptote y = 3



Type II: $C < 0$

C = –4, Asymptotes: y = 3, x = –2, x = 2



4. The vertical asymptotes will occur when the denominator $x^2+C=0$. This can only happen when $C<0$ (Type II). Their equations are $x=\sqrt{-C}$ and $x=-\sqrt{-C}$.

5. General solutions
   a.
   $$f(0) = -2$$
   $$-2 = 3 - \frac{2}{(0)^2 + C}$$
   $$C = \frac{2}{5}$$
   $$y = 3 - \frac{2}{x^2 + \frac{2}{5}}$$
   b. $f(-1) = 4; \quad y = 3 - \frac{2}{x^2 - 3}$
   c. $f(3) = 1; \quad y = 3 - \frac{2}{x^2 - 8}$
6. Domains
   a. $-\infty < x < \infty$ or All real numbers.
   b. The domain is the interval between the asymptotes: $-\sqrt{3} < x < \sqrt{3}$ or the open interval $\left(-\sqrt{3}, \sqrt{3}\right)$
   c. The domain is the interval to the right of the right-side asymptote: $x > \sqrt{8}$ or the open interval $\left(\sqrt{8}, \infty\right)$
7. Extreme value
   a. $\frac{dy}{dx} = 0$ when *x* = 0 and changes sign from negative to positive there, so by the first derivative test there is a minimum value at *x* = 0.
   b. The minimum point is $\left(0, 3 - \frac{2}{C}\right)$.
   c. If *C* = 0, then $3 - \frac{2}{C}$ is undefined. The $\lim_{x \to 0}\left(3 - \frac{1}{x^2 + 0}\right) = -\infty$, so the graph has the *y*-axis as its single vertical asymptote.
   d. Minimum points
      i. (0, –2)
      ii. (3, 11/3) or (0, 3.667)
      iii. (0, 13/4) or (3.25)

8. Concavity

a. $\frac{d}{dx}\left(x\left(y-3\right)^2\right)=\left(1\right)\left(y-3\right)^2+2x\left(y-3\right)\frac{dy}{dx}$

$$=\left(y-3\right)^2+2x\left(y-3\right)\left(x\left(y-3\right)^2\right)$$

$$=\left(y-3\right)^2\left(1+2x^2y-6x^2\right)$$

b. At (0, –2), $\frac{d^2y}{dx^2}=\left(-2-3\right)^2\left(1+2(0^2)(-2)-6(0)^2\right)=25$. Therefore, concave up.

At (–1, 4), $\frac{d^2y}{dx^2}=\left(4-3\right)^2\left(1+2(-1)^2(4)-6(-1)^2\right)=3$. Therefore, concave up.

At (3, 1), $\frac{d^2y}{dx^2}=\left(1-3\right)^2\left(1+2\left(3^2\right)(1)-6\left(3^2\right)\right)=-140$. Therefore, concave down.

These all agree with the graph.

9. The special case $y=3$

a. Since $\frac{dy}{dx}=0$ substituting gives $0=x\left(3-3\right)$ which checks.

b. Using the point (4,3) as an initial point: $3=3-\frac{2}{4^2+C}$ simplifies to $0=-\frac{2}{16+C}$. this equation has no solution.

c. $\lim_{C\to\infty}\left(3-\frac{2}{x^2+C}\right)=3$ and $\lim_{C\to-\infty}\left(3-\frac{2}{x^2+C}\right)=3$

d. In the original solution, to separate the variables it is necessary to divide by $\left(y-3\right)^2$. This is only permissible if $\left(y-3\right)^2\neq 0$ or $y\neq 3$.