Hilbert’s Hotel

It is early in a student’s math life that the concept of infinity is mentioned. Usually it comes up when stating that the counting numbers go on forever. Later, when the behavior of curves is introduced, we explain that a graph going to infinity means that it goes up forever and going to negative infinity means that it goes down forever.

It isn’t until calculus that some perceptions that students have about infinity are not accurate. For instance, in examining $\int_{-1}^{1} \frac{1}{x} \, dx$, students reason that while both areas are infinite, the fact that one is above the $x$-axis and one below the $x$-axis means that they cancel each other out and $\int_{-1}^{1} \frac{1}{x} \, dx = \infty - \infty = 0$. This, of course is not true and it takes a lot of explaining to convince some that $\int_{-1}^{1} \frac{1}{x} \, dx$ does not exist. I always tell students that when you start playing around with infinity, all the rules you learned for numbers go out the window because infinity is not a number. And strange things start occurring.

I like to tackle the calculus problem above early in a class period because I like to have some time to tell them a story. It is a famous math problem but I embellish it a bit to make it sound more like my own.

A man goes to a hotel looking for a room. The hotel has 200 rooms. He goes up to the desk and the conversation with the clerk goes like this:

“I’d like a room please.”

“Sorry, we’re all filled up.”

“What do you mean you’re all filled up? What about room 1?”

“Filled.”

“What about room 27?”

“Filled.”

“What about room 200?”

“Filled. Sir, whatever room you ask for, I am going to tell you it is filled. There is no room for you.”

“Can’t you rearrange some people to get me a room?”
“Sir, no matter how I rearrange rooms, there are 200 guests and 200 rooms and there will be one guest to each room and there will be no room for you. I have an idea. Go across the street and there is another hotel. I feel sure they can help you.”

So the man goes across the street and sees Hilbert’s Hotel that advertises that it has an infinite number of rooms. But he also sees a sign outside that says “no vacancies.” Still, he goes up to the desk and the conversation goes like this:

“I’d like a room please.”

“Sorry, we’re all filled up.”

“What do you mean you’re all filled up? You have an infinite number of rooms. What about room 1?”

“Filled.”

“What about room 2?”

“Filled.”

“What about room 27?”

“Filled.”

“What about room 36,528,997?”

“Filled. Sir whatever room you ask for, I am going to tell you it is filled. For any value of n, room n is filled.”

The man thought about it and said, “I know how you can give me a room. Take the person in room 1 and put him in room 2, the person in room 2 goes to room 3, the person in room 3 goes to room 4, etc. The person in room n goes to room n + 1. Then you can give me room 1.”

And thus the legend of Hilbert’s Hotel was born – the hotel that, even if it is filled, always has a vacancy. This is a famous math problem in logic introduced by German mathematician David Hilbert in a 1924 lecture. There are some interesting variations on Hilbert’s Hotel. For instance:

• If 1 million people show up at the hotel wanting a room, we move the occupant of room 1 to room 1,000,001, room 2 to 1,000,002, etc. That leaves the first million rooms available for the new guests. So for any finite number of guests who show up, they can be put up in Hilbert’s Hotel.

• Now suppose an infinite number of guests shows up to Hilbert’s Hotel. Can they be accommodated? Sure can. Move the occupants of room 1 to room 2, room 2 to room 4, room 3 to room 6, and so on. Room n moves to room 2n. So now all the even number rooms are occupied and the new infinite number of guests can be placed in the odd number rooms.
• Suppose the hotel is filled with an infinite number of people on tour A. A bus with an infinite number of people shows up on tour B and another bus with an infinite number of people shows up on tour C. We want to accommodate them all, but since they all get wake-up calls at different times, we need to easily define the rooms for tour A, tour B, and tour C.

We start by moving the occupants of room 1 to room 2, room 2 to room 4, room 3 to room 6, and so on. Room $n$ moves to room $2n$. That means tour A is in the even number rooms. So all the odd number rooms are free. We place members of tour B in rooms 3, 5, 7, 11, 13,… So tour B is in all the odd number prime rooms. That leaves tour C for the odd number non-prime rooms. So room 128 contains a tour A guest, room 127 contains a tour B person, and room 129 contains a tour C person.

• Now suppose an infinite number of buses show up, each with an infinite number of guests. We will denote the bus number as $b$ and the passenger’s seat number on the bus to be $n$. Can they all be accommodated? No problem. We again start by moving the occupants of room 1 to room 2, room 2 to room 4, room 3 to room 6, and so on. Room $n$ moves to room $2n$. So all the odd number rooms are free.

Place all the first bus passengers in rooms 3, 9, 27, 81,…, $3^n$. Place all the second bus passengers in rooms 5, 25, 125, 625,…, $5^n$, all the third bus passengers in rooms 7, 49, 343, 2401,…, $7^n$, and so on. Bus $b$ uses rooms $p, p^2, p^3,…, p^n$ where $p$ is the $b^{th}$ prime number. This solution actually leaves rooms like 21 and 255 that are not prime powers unoccupied. How many unoccupied rooms are there? You guessed it: an infinite number of rooms. Having to find rooms for the infinitely many buses with infinitely many passengers each, actually managed to free up an infinitely many extra rooms. Yes, this is a strange hotel!

• Now, suppose an infinite number of ships dock next to the hotel, each with an infinite number of buses, each with an infinite number of passengers. Yes, they can all be accommodated. We again move the room $n$ to room $2n$ leaving all the odd rooms available. The passenger in the 2nd ship, 3rd bus, 3rd seat would raise the 2nd odd prime (5) to the 3rd odd prime power (7) raised to the 3rd power: $5^7^3$ has over 239 digits.

• Suppose the Hilbert Hotel does some expansion and places an infinite number of rooms between room 1 and room 2, an infinite number of rooms between room 2 and room 3, etc. If the hotel was full, surely it could not accommodate one more person right?

Wrong. Think of the rooms as fractions. $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4},…$ will be all the rooms between one inclusive and two. $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4},…$ will be all the rooms between 2 inclusive and 3, etc. So when that one extra person comes along, we move people from room to room using the method shown by the arrows in the figure to the right. That leaves room 1/1 for the new guest. This suggests that there the same number of whole numbers as fractions, even though both are infinite.

There’s always room at the Hilbert Hotel!

When you show this to students, you will meet with resistance from some. They cannot grasp that all the rooms will be filled. They believe that no matter how many are filled, there are always infinitely more than are empty. But it is possible for the hotel to be completely filled yet still have room for one more or an infinite number more. It is the concept of “fullness” that leads their intuition astray.
In the finite world, a hotel that is full means that there is room for no more and rearranging will make no difference. In the infinite world, the hotel that is full still has room just by rearranging guests. This is logically consistent but certainly counterintuitive and paradoxical and some students cannot grasp it. I have found that the students who fight this problem are sometimes the deepest thinkers. I enjoy the debate with other students trying to convince them about the feasibility of the problem.

There is a lot of theory in this fun problem and if you show it to students, suggest that they go on-line and look up Georg Cantor, a German mathematician who designed set theory and made sense of the infinite types of infinities. Cantor started the use of Cardinal numbers to differentiate the different types of infinities (for instance the cardinal number of whole numbers is the same as the cardinal number of rational numbers).

Infinity fascinates students. They are told relatively early in life that space is infinite. Our brains have a lot of trouble grasping that. There is a finite amount of water on the earth, a finite number of grains of sand, a finite amount of air. And yet these numbers are so large as to be uncountable. It is impossible to conceive that space goes on and on. And yet, if there were an end to outer space, what would it look like? Would there be a wall? If so, what would be on the other side of the wall?

It has always been my theory that space is curved. Just as the earth is curved and if you move in the same direction, you will eventually come back to the same spot. I believe that is true with space: if you take off from earth and continue in the same direction, you will eventually come back to the earth (although this is complicated by the fact that the earth is moving through the solar system and space). I like to throw out these theories to students. It gets them to think about infinity and grasp the ungraspable. Note that this can move to the role of God and infinity (“God is infinite.”). Depending on your school district, you would be wise to deflect this type of discussion. It is sure to get reported to a parent and all of a sudden, you are asked by the principal to defend your discussing religion in math class. You just don’t want to go there!

You can show the Hilbert Hotel problem at any time as a fun diversion to get students to think about infinity. But I believe it is somewhat essential before teaching L’Hospital’s rule for the indeterminate forms $\infty - \infty, 1^\infty,$ and $\infty^0$ to let students see that all is not as it seems. Above all, it gets them to treat infinity with a healthy respect!

Want to make a comment? Send me an email at Divinginmmm@gmail.com.