Lin McMullin

How to Solve Any Triangle: First, Forget the Law of Sines and the Law of Cosines

Computer algebra systems (CASs) have made, and will continue to make, changes in what is taught in mathematics courses at all levels. Just as scientific calculators made teaching computations with logarithms obsolete, CASs will render obsolete many other topics in the curriculum. Some will be formulas and procedures for computations that are no longer necessary to solve the problems for which they were originally developed. The mathematical problems will remain; we will just use more efficient ways to solve them. A good example is Newton's method for approximating the zeros of an expression. Using Newton's method to find roots by hand has been rendered obsolete by graphing calculators that can either find the roots exactly or do the approximations faster and more accurately.

Two other pillars of trigonometric computation, the law of sines and the law of cosines, can now be retired. They were used for "solving a triangle," when a few of its parts (sides and angles) were given. The law of sines and the law of cosines are used to make computation by hand as easy as possible. With the advent of CASs, ease of computation is no longer a consideration. Now we can concentrate on ease of method. The distance formula lends itself to solving triangles. The equations that are suggested here are easy to write but difficult to solve by hand. With a CAS, we need to be able to write an equation or a system of equations. The CAS will solve them.

The law of sines and the law of cosines can now be retired

A set of typical triangle-solving problems follow. Each of the examples uses a triangle labeled as shown in **figure 1.** The SSA, SAS, and SSS cases use the distance formula to write an equation relating the distance from B to C to the givens and unknowns in the problem; the ASA case is different but easier.

In **figure 1,** the coordinates of B are (c cos A, c sin A); they are also ($b - a$ cos C, a sin C). If A is an obtuse angle, then B will be in the second quadrant, but the general coordinates of B will be the same.

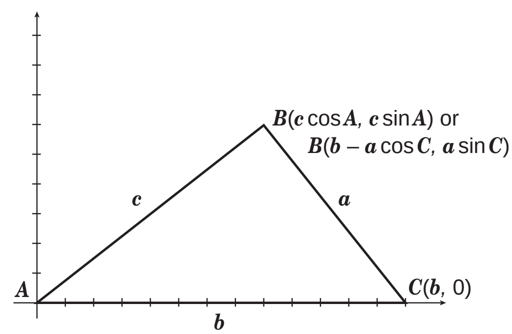

Fig. 1

THE SSA CASES

1. Given $m\angle A = 38$, $a = 8$, and $c = 10$, we can find b by using the distance formula to write an equation for $BC = a$:

$$(10 \cos 38 - b)^2 + (10 \sin 38 - 0)^2 = 8^2$$

The CAS gives two solutions: $b = 12.989$ and $b = 2.772$. This problem is a two-solution law-of-sines problem, and the CAS automatically gives both answers. Determining the number of solutions requires no additional analysis.

2. If instead, $a = 12$, the equation becomes

$$(10 \cos 38 - b)^2 + (10 \sin 38)^2 = 12^2.$$

The solutions by CAS are $b = 18.180$ and $b = -2.420$. Since -2.420 is negative, it is obviously not possible, so it is rejected.

3. If $a = 3$, the equation becomes

$$(10 \cos 38 - b)^2 + (10 \sin 38)^2 = 3^2.$$

The CAS tells us that this equation is false; therefore, no solution exists.

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The CAS has handled all the so-called ambiguous cases with ease.

THE SSS CASE

Given $a = 5$, $b = 10$, and $c = 8$, we can find the measure of angle A by again using the distance formula to write the equation concerning the length of $BC = a$:

$$(8 \cos A - 10)^2 + (8 \sin A)^2 = 5^2,\ 0 < A < 180$$

The solution by CAS is *m*–*A* = 29.686.

THE SAS CASE

Given $b = 16$, $c = 5$, and *m*–*A* = 38, we can find a by again using the distance formula to write the equation

$$(5 \cos 38 - 16)^2 + (5 \sin 38)^2 = a^2.$$

Again rejecting a negative solution, we find that the solution by CAS is $a = 12.447$.

THE ASA CASE

Given *m*– *A* = 38, *m*– *C* = 27, and $b = 10$, the pattern here is different because it does not use the distance formula. We can write the coordinates of B in two different ways. They are $(c \cos A, c \sin A)$ and $(b - a \cos C, a \sin C)$. We equate the x-coordinates and the y-coordinates and solve the resulting system of equations using a CAS:

$$c \cos 38 = 10 - a \cos 27$$
$$c \sin 38 = a \sin 27$$

The solutions are $a = 6.793$ and $c = 5.009$.

We can deal with all the solution possibilities by knowing only the general coordinates of B and the distance formula, both of which students need to know for other situations. These requirements are obviously less and, in my opinion, easier than learning the law of sines with its ambiguous cases and the two forms of the law of cosines (one for sides and one for angles), and knowing which of the laws to use for a given problem.

Such pillars of the curriculum do not topple easily. They do have an intrinsic beauty and elegance. The utility of the two laws comes from the fact that if the triangles are to be solved by hand, they provide the easiest computational scheme. The techniques suggested here produce equations that are more difficult to solve by hand. However, with CASs, we no longer must solve equations by hand.

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