The Golden Ratio and Quartic Polynomials

By Lin McMullin

The Golden Ratio is a number that was first observed in antiquity and has been tuning up ever since sometimes in the most unexpected places. I certainly did not expect it to appear in a problem I was investigating involving quartic (fourth degree) polynomials.

The Golden Ratio

The Golden ration is defined as the ratio determined by dividing a segment in such a way that the shorter part is to the loner part as the longer part is to the entire segment. (See Figure 1)

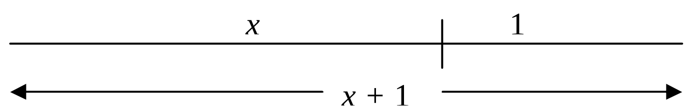

Figure 1

If the shorter part is 1 unit and the longer part is x units then the proportion is

$$\frac{1}{x} = \frac{x}{x+1}$$
$$x^2 = x+1$$
$$x^2 - x - 1 = 0$$
$$x = \frac{1+\sqrt{5}}{2} \text{ and } x = \frac{1-\sqrt{5}}{2}$$

The first solution is called the Golden Ratio and is usually denoted by $\Phi = \frac{1+\sqrt{5}}{2}$ (called "Big Phi") and its conjugate is denoted by $\varphi = \frac{1-\sqrt{5}}{2}$ ("Little Phi").

A problem with quartic polynomials

A fourth degree polynomial may have a graph that is shaped somewhat like a “W” (or an “M”). If so, then there are two points of inflection, points where the concavity of the graph changes. I was interested in finding the other two points in which the line through the points of inflection intersects the graph. These other two points will be called x_L and x_R (See Figure 2).

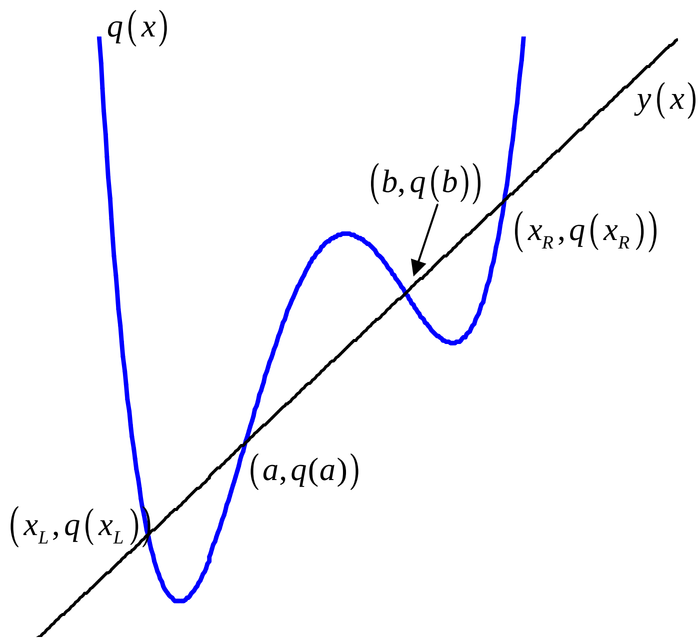

Figure 2

One way to approach the problem would be to start with the general fourth-degree polynomial:

$$q(x) = m_4x^4 + m_3x^3 + m_2x^2 + m_1x + m_0 \tag{1}$$

Differentiate this twice and find the roots of the second derivative; then use these to write the equation of the line. These are somewhat complicated so I used a different approach without which the appearance of the Golden Ratio would not have been at all obvious. I began by letting *a* and *b* be the roots of the second derivative.

$$q''(x) = 12m_4(x-a)(x-b) = 12m_4x^2 - 12m_4(a+b)x + 12m_4ab \qquad (2)$$

Integrating this twice, using first m_1 and then m_0 as the constants of integration, produced $q(x)$ in a slightly different form.

$$q'(x) = 4m_4x^3 - 6m_4(a+b)x^2 + 12m_4abx + m_1$$

$$q(x) = m_4x^4 - 2m_4(a+b)x^3 + 6m_4abx^2 + m_1x + m_0 \qquad (3)$$

I then used this to write the equation of the line through the points of inflection.

$$\begin{aligned} y(x) &= \frac{q(b)-q(a)}{b-a}(x-a) + q(a) \\ &= -\left(m_4a^3 - 3m_4a^2b - 3m_4ab^2 + m_4b^3 - m_1\right)x + (m_4a^3b - 3m_4a^2b^2 + m_4ab^3 + m_0) \end{aligned}$$

$$(4)$$

Finally, I solved the equation $q(x) = y(x)$ for the four points of intersection. Two of the solutions are, of course, $x = a$ and $x = b$. The other two are x_L and x_R where

$$\begin{aligned} x_L &= \left(\frac{1+\sqrt{5}}{2}\right)a + \left(\frac{1-\sqrt{5}}{2}\right)b = \Phi a + \varphi b \\ x_R &= \left(\frac{1+\sqrt{5}}{2}\right)b + \left(\frac{1-\sqrt{5}}{2}\right)a = \Phi b + \varphi a \end{aligned} \qquad (5)$$

(If $a < b$, then the numbers in order from left to right are $x_L < a < b < x_R$.)

The computations were done using a Texas Instruments Voyage 200 calculator with its built in CAS and duplicated using TI-Interactive. (See sidebars 1 and 2)

Quartics without points of inflection

If $a = b$ then the quartic will not have any point of inflection. The result still holds with $x_L = a = b = x_R$.

However, the computation made no assumptions about *a*, *b*, or the coefficients $m_0, m_1, ..., m_4$; any or all of them could be Complex numbers and the results (5) still are still true. In this case the quartic has no points of inflection (it shape is similar to a parabola), but nevertheless the solutions of the equation $q(x) = y(x)$ have the relationship expressed in (5).

In general, for any quartic polynomial $q(x)$, if $x = a$ and $x = b$ are the roots of $q''(x) = 0$ and are usedto produce the linear function $y(x) = \frac{q(b)-q(a)}{b-a}(x-a)+q(a)$, then $q(x) = y(x)$ for the numbers $x = a$ and $x = b$ and

$$x_L = \left(\frac{1+\sqrt{5}}{2}\right)a + \left(\frac{1-\sqrt{5}}{2}\right)b = \Phi a + \varphi b$$
$$x_R = \left(\frac{1+\sqrt{5}}{2}\right)b + \left(\frac{1-\sqrt{5}}{2}\right)a = \Phi b + \varphi a$$

The Golden Ratio, Φ, and its conjugate, φ, are lurkng in every quartic polynomial.

Sidebar 1: The CAS computation done on a V 200

$$\blacksquare \int\int(12\cdot m4\cdot(x-a)\cdot(x-b))\,dx\,dx$$

$$m4\cdot x^4-2\cdot(a+b)\cdot m4\cdot x^3+6\cdot a\cdot b\cdot m4\cdot x^2+m1\cdot x+m0$$

$$\blacksquare\; m4\cdot x^4-2\cdot(a+b)\cdot m4\cdot x^3+6\cdot a\cdot b\cdot m4\cdot x^2+m1\cdot x+m0\rightarrow q(x)$$

Done

$$\blacksquare\; \frac{q(b)-q(a)}{b-a}\cdot(x-a)+q(a)$$

$$-\left(a^3\cdot m4-3\cdot a^2\cdot b\cdot m4-3\cdot a\cdot b^2\cdot m4+b^3\cdot m4-m1\right)\cdot x+a^3\cdot b\cdot m4-3\cdot a^2\cdot b^2\cdot m4+a\cdot b^3\cdot m4+m0$$

$$\blacksquare\; -\left(a^3\cdot m4-3\cdot a^2\cdot b\cdot m4-3\cdot a\cdot b^2\cdot m4+b^3\cdot m4-m1\right)\cdot x+a^3\cdot b\cdot m4-3\cdot a^2\cdot b^2\cdot m4+a\cdot b^3\cdot m4+m0\rightarrow y(x)$$

Done

$$\blacksquare\; \text{solve}(q(x)=y(x),x)$$

$$x=\frac{a\cdot(\sqrt{5}+1)-b\cdot(\sqrt{5}-1)}{2}\text{ or }x=\frac{-\left(a\cdot(\sqrt{5}-1)-b\cdot(\sqrt{5}+1)\right)}{2}\text{ or }x=a\text{ or }x=b$$

Each line begins at the ■ – the result is the next line.

- Line 1: The nested integration with initial conditions is entered as
`∫(∫(12*m4*(x-a)*(x-b).x.m1).x.m0)` and returns the expression for $q(x)$.
- Line 2: The expression is stored to $q(x)$
- Line 3: Entering the point-slope form of the equation of the line through $(a,q(a))$ and $(b,q(b))$ returns the equation in $mx+b$ form.
- Line 4: The equation of the line is stored to $y(x)$.
- Line 5: Finding the coordinates of the 4 points of intersection by solving $q(x)=y(x)$ for *x*.

End Figure 3

Sidebar 2: The computation done on TI-Interactive

$$\int (12 \cdot m4 \cdot (x-a) \cdot (x-b))\, dx + m1$$

$$4 \cdot m4 \cdot x^3 - 6 \cdot (a+b) \cdot m4 \cdot x^2 + 12 \cdot a \cdot b \cdot m4 \cdot x + m1$$

$$\int ans\, dx + m0$$

$$m4 \cdot x^4 - 2 \cdot (a+b) \cdot m4 \cdot x^3 + 6 \cdot a \cdot b \cdot m4 \cdot x^2 + m1 \cdot x + m0$$

$$ans \rightarrow q(x)$$

"Done"

$$\frac{q(a) - q(b)}{a-b} \cdot (x-a) + q(a)$$

$$-(a^3 \cdot m4 - 3 \cdot a^2 \cdot b \cdot m4 - 3 \cdot a \cdot b^2 \cdot m4 + b^3 \cdot m4 - m1) \cdot x + a^3 \cdot b \cdot m4 - 3 \cdot a^2 \cdot b^2 \cdot m4 + a \cdot b^3 \cdot m4 + m0$$

$$ans \rightarrow y(x)$$

"Done"

$$\text{solve}(q(x) = y(x), x)$$

$$x = \frac{a \cdot (\sqrt{5}+1) - b \cdot (\sqrt{5}-1)}{2} \text{ or } x = \frac{-(a \cdot (\sqrt{5}-1) - b \cdot (\sqrt{5}+1))}{2} \text{ or } x = a \text{ or } x = b$$

End Figure 4