Questions for 2007 BC Calculus Institutes

CALCULATOR

1. How many zeros does the function $f(x) = \sin(\ln(x))$ have in the interval (0,1]?

   Explain how you know.

LIMITS

2. <u>2003 Released Exam Limit Questions</u> 3, 6, 79

3. $\lim\limits_{x\to\infty} \dfrac{\ln(x^5)}{x^{0.02}} =$

4. How many times do the graphs of $y = 2^x$ and $y = x^4$ intersect?

5. 1998 AB 2

   Let f be the function given by $f(x) = 2xe^{2x}$.
   (a) Find $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$.
   (b) Find the absolute minimum value of f. Justify that your answer is an absolute minimum.
   (c) What is the range of f?
   (d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

DERIVATIVES

6. $\operatorname{limit}\limits_{h\to 0} \dfrac{3\sin(\frac{\pi}{7}+h) - 3\sin(\frac{\pi}{7})}{h} =$

   (A) 0 (B) 0.434 (C) 0.901 (D) 1.302 (E) 2.703

7. $\operatorname{limit}\limits_{h\to 0} \dfrac{\cos\left(\frac{3\pi}{2}+h\right)}{h} =$

(A) Nonexistent (B) –1 (C) 0 (D) $\frac{1}{h}$ (E) 1

8. 2003 Released Exam 2,

9. 2003 Released Exam Derivative theory and MVT

    a. AB: 13, 16, 80,

    b. BC: 16, 19, 27,83,92 and

    c. 2005 AB 3(d) a very "formal" use of the MVT but one your students should see.

10. 2003 Released Exam Computing Derivatives:

    a. Basic rules AB: 1, 4, 9, 14, BC: 1, 9, 17

    b. Inverses AB: 27, BC: 27,

    c. FTC: AB 23, 92, BC: 18

    d. Implicit: AB 26, also 2004 AB4/BC4

    e. BC 1, 9, 23, **79** (compare with 2007 AB free-response 3 where $\bar{x} = 0.95$ out of 9 points)

11. *TAPC* 2/e p. 72 – 73 Table and see 2007 AB 3 ($\bar{x} = 0.95$ out of 9 points. See 2003 BC 79); 2006 AB 6

12. What is the value of the derivative of the inverse of the function $y = e^x - e^{-x}$ at (0,0)?

DERIVATIVE APPPLICATIONS

13. 1995 BC 5 (Suitable for AB)

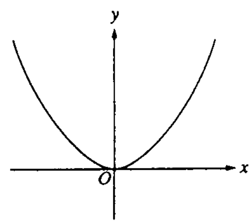


Figure 1
$y = f(x)$

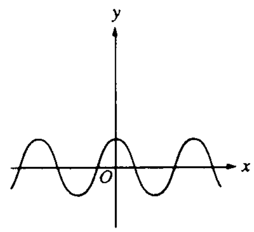


Figure 2
$y = g(x)$

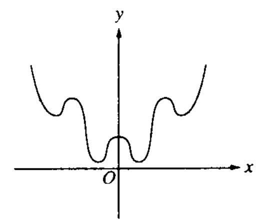


Figure 3

Let $f(x) = x^2$, $g(x) = \cos x$, and $h(x) = x^2 + \cos x$. From the graphs of f and g shown above in Fig. 1 and Fig. 2, one might think the graph of h should look like the graph in Fig.3.

(a)Sketch the actual graph of h in the viewing window provided below, $(-6 \le x \le 6$ and $-6 \le y \le 40)$.

(b)Use $h''(x)$ to explain why the graph of h does not look like the graph in Fig 3.

(c)Prove that the graph of $y = x^2 + \cos(kx)$ has either no points of inflection or infinitely many points of inflection, depending on the value of the constant *k*.

14. Consider the function $f(x) = x^3 - 3x^2 - 24x + k$

(a)Find the x-coordinate of the function's relative maximum. Justify your answer.

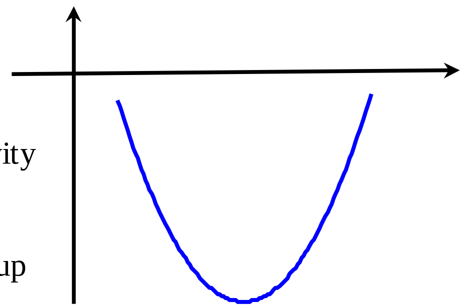
(b) Find the x-coordinate of the function's relative minimum. Justify your answer.

(c)Find the x-coordinate of the function's point of inflection.

(d) Given that *f* has exactly 2 real roots, find <u>both</u> possible values of k.

15. 2007 Form B AB 4 and many others. See Type Question handout. 31

16. The figure shows a small section of the graph of the derivative of a function. Which of the choices *best* describes the corresponding part of the graph of the function?



a. Decreasing and concave up only
b. Decreases and changes from concave down on the left to up on the right.
c. Decreases and does not change concavity
d. Increasing and concave up
e. Increasing and changes from concave up on the left to down on the right.

17. 2003 Released Exam BC 12, 78, 82, 86, 87, 90, 91

OPTIMIZATION

18. A wire 3 feet long is cut and formed into a square and a circle. Where should the wire be cut so that the total area of the square and a circle is a maximum?

19. 1982 AB 6: A tank with a rectangular bottom and rectangular sides is to be open at the top. It is constructed so its width is 4 meters and its volume is 36 cubic meters. If building the tank costs $10 per square meter for the bottom and $5 per square meter for the sides, what is the cost of the least expensive tank?

INTEGRATION

20. **Exploration 1:** A tank is being filled with water using a pump that is old, and slows down as it runs. The table below gives the rate at which the pump pumps at ten-minute intervals. If the tank is initially empty, approximate how much water is in the tank after 90 minutes?

Elapsed time (Minutes)	0	10	20	30	40	50	60	70	80	90
Rate (gallons / minute)	42	40	38	35	35	32	28	20	19	10

See notes in *TAPC 2/e* p. 103-104

Exploration 2: The speed of an airplane in miles per hour is given at half-hour intervals in the table below. Approximately, how far does the airplane travel in the three hours given in the table? How far is it from the airport?

Elapsed time (minutes)	0	30	60	90	120	150	180
Speed (miles per hour)	375	390	400	390	385	350	345

See notes in *TAPC 2/e* p. 103-104

Riemann sums

21. Approximate $\int_1^4 1+x^2dx$ using (1) a left Riemann sum with 6 equal subdivisions and (2) a right Riemann sum with 6 equal subdivisions.

22. 2003 Released Exam Riemann sums AB 85; BC 18, 25, 85, 88;

23. Let T be any Trapezoidal rule approximation to $S = \int_a^b 3x - x^3dx$. Which statement is true?

    I. If $a < b < 0$, then $T > S$.

    II. If $a < 0 < b$, then $T = S$.

    III. If $0 < a < b$, then $T < S$.

    (A) I only II. II only (C) III only (D) I and III only (E) I, II and III

24. Let f be a continuous function defined for all x such that $3 \le f(x) \le 7$. The largest possible value for any Riemann sum for f on the interval [1,5] is

    (A) 3 (B) 7 (C) 12 (D) 16 (E) 28

25. (1997 AB 24) The expression $\frac{1}{50}\left[\sqrt{\frac{1}{50}}+\sqrt{\frac{2}{50}}+\sqrt{\frac{3}{50}}+\cdots+\sqrt{\frac{50}{50}}\right]$ is a Riemann Sum for

    (A) $\int_0^1 \sqrt{\frac{x}{50}}\,dx$ (B) $\int_0^1 \sqrt{x}\,dx$ (C) $\frac{1}{50}\int_0^1 \sqrt{\frac{x}{50}}\,dx$

(D) $\frac{1}{50}\int_0^1 \sqrt{x}\,dx$ (E) $\frac{1}{50}\int_0^{50} \sqrt{x}\,dx$

26. $\underset{n\to\infty}{\text{limit}} \sum_{k=1}^{n} e^{\left(\frac{2k}{n}\right)}\left(\frac{2}{n}\right) =$

27. $\underset{n\to\infty}{\text{limit}} \sum_{k=1}^{n} \left[2+\frac{3}{n}k\right]^2\left(\frac{3}{n}\right) =$

28. If the closed interval [0, b] is divided into equal parts each of length $\frac{b}{n}$, then $\int_0^b f'(x)\,dx =$

I. $f(b)$ II. $\underset{n\to\infty}{\text{limit}} \sum_{k=1}^{n} \left(f'\left(\frac{b}{n}k\right)\right)\left(\frac{b}{n}\right)$ III. $f(b) - f(0)$

(A) I only (B) II only (C) III only (D) I and III only (E) II and III only.

29. If t is measured in hours and $f'(t)$ is measured in knots, what is the value of $\int_0^2 f'(t)dt$? (Note: 1 knot = 1 nautical mile per hour)

(A) $f(2)$ knots (B) $f(2) - f(0)$ knots (C) $f(2)$ nautical miles

(D) $f(2) - f(0)$ nautical miles (E) $f(2) - f(0)$ knots per hour.

30. 1998 AB #88: Let $F(x)$ be an antiderivative of $\frac{(\ln x)^3}{x}$. If $F(1) = 0$, then $F(9) =$

(A) 0.048 (B) 0.144 (C) 5.827 (D) 23.308 (E) 1,640.250

31. The table below gives the velocity in the vertical direction of a rider on a Ferris wheel at an amusement park. The rider moves smoothly and the table gives the values for one complete revolution of the wheel. (This is similar to 1998 AB 3)

a. During what interval of time is the acceleration negative? Give a reason for your answer.
b. What is the average acceleration during the first 15 seconds of the ride? Include units of measure.
c. Approximate $\int_0^{30} v(t)dt$ using a Riemann Sum with six intervals of equal length.
d. Approximate the diameter of the Ferris Wheel. Explain your reasoning.

t seconds	v feet/second
0	0
5	1.6
10	2.7
15	3.1
20	2.7
25	1.6
30	0
35	-1.6
40	-2.7
45	-3.1
50	-2.7
55	-1.6
60	0

PARAMETRIC AND POLAR EQUATIONS

32. From the 2003 Released Exam BC 4, 7, 15, 17, 84

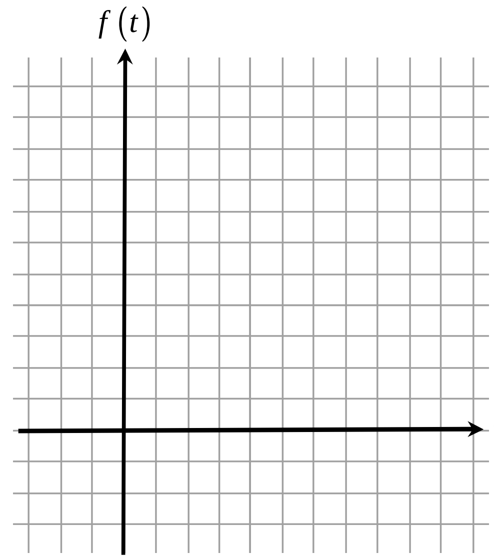
FR 2003 BC 3, 2007 BC 3

ACCUMULATION

33. Investigation 2:

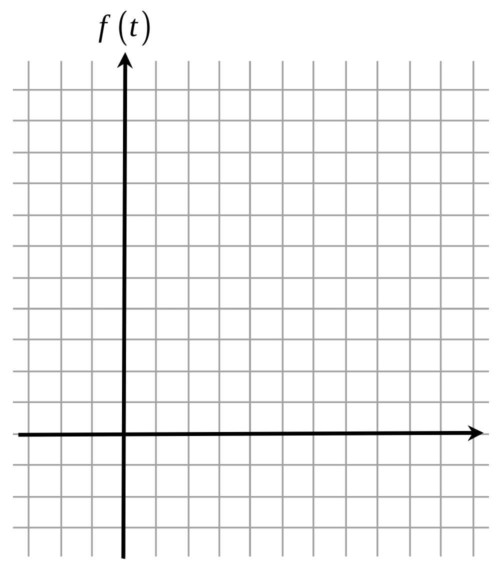
a. On the axes provided graph $f\left(t\right)=3$.

Let $[0,x]$, be an interval on the *t*-axis. Write the equation of the function $A_1(x)$ that gives the area of the region in the first quadrant under the graph of $y=f(t)$, above the *t*-axis, between t = 0 and t = x. Indicate where this region appears on the graph by shading a typical region and indicating where x is.

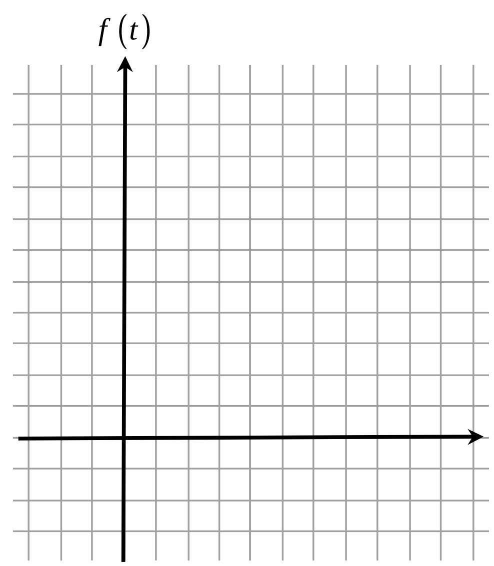


b. On the axes provided graph $f\left(t\right)=2t$.

Let $[0,x]$, be an interval on the *t*-axis. Write the equation of the function $A_2(x)$ that gives the area of the region in the first quadrant under the graph of $y=f(t)$, above the *t*-axis, between t = 0 and t = x. Indicate where this region appears on the graph by shading a typical region and indicating where x is.



c. On the axes provided graph $f\left(t\right)=2t+3$. Let $[0,x]$, be an interval on the *t*-axis. Write the equation of the function $A_3(x)$ that gives the area of the region in the first quadrant under the graph of $y=f(t)$, above the *t*-axis, between t = 0 and t = x. Indicate where this region appears on the graph by shading a typical region and indicating where x is.



d. Fill in the table for these functions

x	0	1	2	3	4	5
$A_1(x)$						
$A_2(x)$						
$A_3(x)$						

Do these numbers agree with your idea of area? Why does $A_3 = A_1 + A_2$? Show graphically why this is true.

e. Fill in the table for these values:

x	–1	–2	–3
$A_1(x)$			
$A_2(x)$			
$A_3(x)$			

Explain your reasoning; specifically tell how does this relates to the area?

f. Calculate:

$\frac{dA_1(x)}{dx} =$; $\frac{dA_2(x)}{dx} =$; $\frac{dA_3(x)}{dx} =$

What do you observe about the derivatives? Why do you think this is?

g. Consider a new function $A_4(x)$ that gives the area under $y = 2t + 3$ on the interval $[2, x]$. Complete the table below and find $\frac{dA_4(x)}{dx}$. Why does $\frac{dA_4(x)}{dx} = \frac{dA_3(x)}{dx}$?

x	–2	–1	0	1	2	3	4	5
$A_4(x)$								

See *TAPC* 2/e p. 116 – 119

34. On the interval $[0, 2\pi]$ which function has an average value that is *not* 0?

I. $\cos(x)$ II. $\sin(\pi x)$ III. $\pi - x$

35. Let *f* and *g* be continuous functions with $f(x) - g(x) = 3$. Which statement is true?

I. On the interval [0, 10] the average value of *f* is 30 more than the average value of *g* .

II. On the interval [0,10] the average value of *g* is 3 less than the average value of *f*.

III. $\int_5^6 f(x)dx - \int_5^6 g(x)\,dx = 3$

(A) I only (B) II only (C) III only (D) I and III only. (E) II and III only

36. <u>2003 Released Exam</u> Applications of integrals AB 82, 84, 86, 88; BC 15, 80, 82, 88, 89.

37. <u>2003 Released Exam</u>: Methods of integration AB 2, 5, 8, 11 and BC 3, 8, 23, 26

DIFFERENTIAL EQUATIONS

38. Slope Fields from past exams; 1998 BC mc:24 and BC4, <u>2000 BC6</u>, <u>2002 BC 5</u>, 2003 BC mc:14, 2004 AB 6, form B AB5, 2005 AB6, <u>BC4</u>, 2006 AB5,

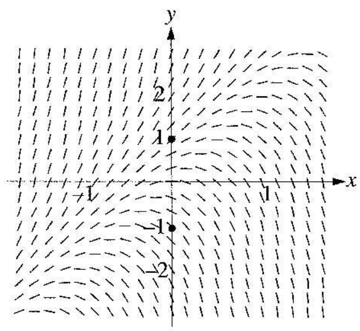
39. Other BC Differential Equation Questions (including Euler's Method) 2003 Released Exam 5, 14, 21, 80.

40. 2005AB6(c) Given $\frac{dy}{dx} = -\frac{2x}{y}$, find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$ (Part (a) was draw a slope field, and part (b) approximate $f(1.1)$ with tangent line at (1, –1).)

41. 2002 AB 5

5. Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point (0, 1) and sketch the solution curve that passes through the point (0, −1).
(Note: Use the slope field provided in the pink test booklet.)



(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point (0, 0) ? If so, is the point a local maximum or a local minimum? Justify your answer.

42. Consider the differential equation $\frac{dy}{dx} = \frac{y - y^2}{x}$ for all $x \neq 0$.

(a) Verify that $y = \frac{x}{x+C}$, $x \neq -C$ is a general solution for the given differential equation.

(b) Write an equation of the particular solution that contains the point (–1, –1) and find the value of $\frac{dy}{dx}$ at (0,0) for this solution.

(c) Write an equation of the vertical and horizontal asymptotes of the particular solution found in (b).

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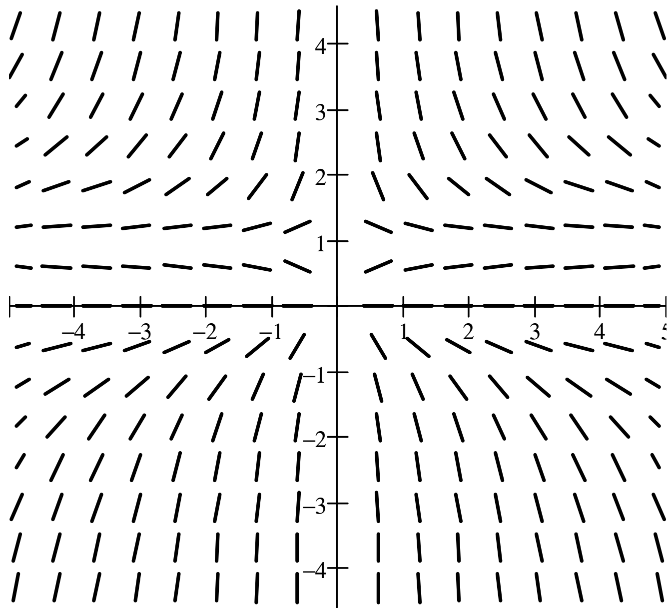
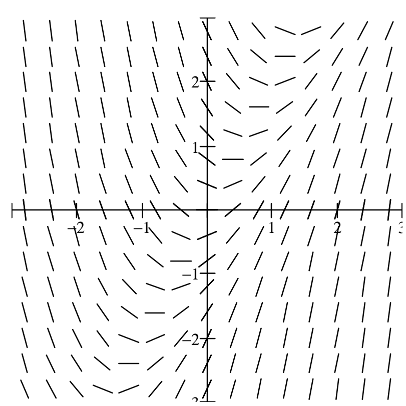
(d) The slope field for the given differential equation is provided. Sketch the particular solution that passes through the point (–1, –1).



(Note: This is a good problem for Winplot. Graph the general solution and use a slider for *C*. Notice the slope at (0,0) is indeterminate, but each solution has a slope there. Also investigate the vertical and horizontal asymptotes and the solution curve when *C* is close to 0.)

43. The slope field for $y' = 2x - y$ is shown below. Which graph could be a solution of the differential equation shown?



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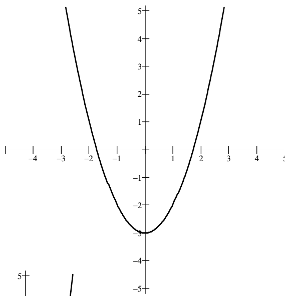
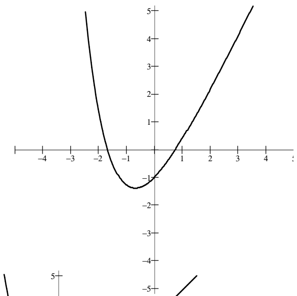
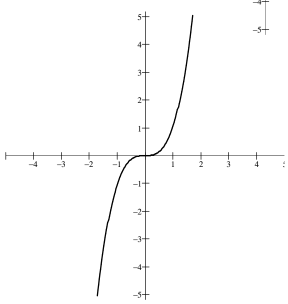
(A)



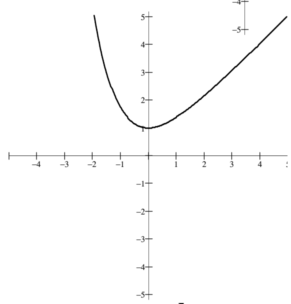
(B)



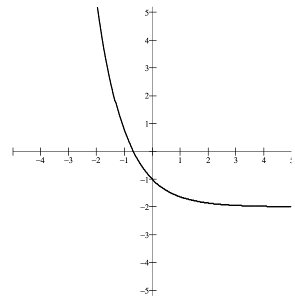
(C)



(D)

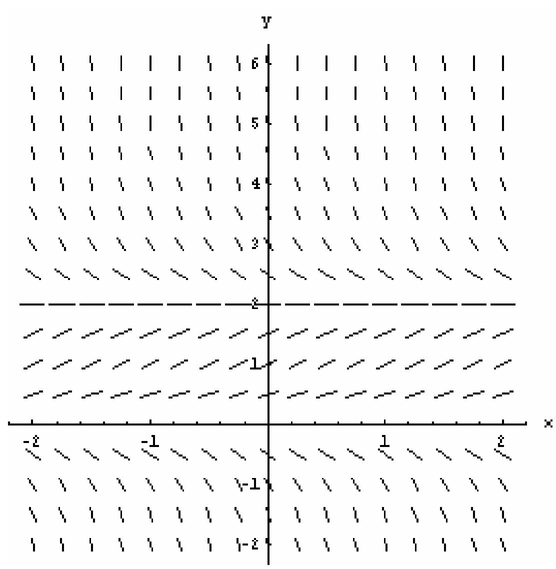


(E)



44. The slope field for a differential equation $\frac{dy}{dx} = f(y)$ is shown in the figure above.

Which statement is true about $y(x)$?



I. If $y(0) > 2$ then $\lim_{x\to\infty} y(x) \approx 2$

II. If $0 < y(0) < 2$ then $\lim_{x\to\infty} y(x) \approx 2$

III. If $y(0) < 2$ then $\lim_{x\to\infty} y(x) \approx 2$

(A) I only (B) II only (C) III only

(D) I and II only (E) I, II and III

45. 2006 AB 5(b): Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$ where $x \neq 0$. Find the particular solution $y = f(x)$ to the differential equation with the initial equation $f(-1) = 1$ and state its domain.

For more on the domain of the solution of a differential equation see the articles by L. Riddle and D. Loman in the Articles File of the Participants' File 2007.

Powers Series Questions

1. Write the first four nonzero terms in the Maclaurin series for xe^{-x}.

2. $\sum_{k=0}^{\infty}\left(-\frac{\pi}{3}\right)^k =$

(A) $\dfrac{1}{1-\frac{\pi}{3}}$ (B) $\dfrac{\frac{\pi}{3}}{1-\frac{\pi}{3}}$ (C) $\dfrac{3}{3+\pi}$ (D) $\dfrac{\pi}{3+\pi}$ (E) The series does not converge.

3. Let E be the error when the Taylor polynomial $T(x)=x-\dfrac{x^3}{3!}$ is used to approximate $f(x)=\sin(x)$ at $x = 0.5$. Which of the following is true?

(A) $|E|<0.0001$ (B) $0.0001<|E|<0.0003$ (C) $0.0003<|E|<0.005$

(D) $0.005<|E|<0.007$ (E) $0.007<|E|$

4. The Taylor series of a function $f(x)$ about $x = 3$ is given by

$$f(x)=1+3(x-3)+\frac{5(x-3)^2}{2!}+\frac{7(x-3)^3}{3!}+\cdots+\frac{(2n+1)(x-3)^n}{n!}$$

What is the value of $f'''(3)$ and $f^{(7)}(3)$?

5. What are all values of x for which the series $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots$ converges?

(A) $-1\le x\le 1$ (B) $-1\le x<1$ (C) $-1<x\le 1$

(D) $-1<x<1$ (E) All real numbers x.

6. $\sum_{k=1}^{n}\dfrac{(-1)^n(\pi)^{2n}}{(2n)!}=$

7. Let $f(x)$ be the function defined by the power series $f(x)=\sum_{k=0}^{\infty}2x^n$. If $g'(x)=f(x)$ and $g(0)=2$ then $g(x)=$

8. Let $f(x)$ be a function with the following properties:

(i) $f(0) = -2$

(ii) $f'(x) = 3f(x)$

(iii) The n^{th} derivative of f, $f^{(n)}(x) = 3f^{(n-1)}(x)$

(a) Give the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Find $f(x)$ by solving the differential equation in (ii) with the initial condition in (i).

(c) Graph and label both f and the third degree Maclaurin polynomial of f on the axes below and label each. [Window is [–2,2] by [–40, 10])

Power Series Answers:

1. $x - x^2 + \frac{1}{2!}x^3 - \frac{1}{4!}x^5 + - \cdots$; **2.** (E); **3.** B; **4.** 7, 15; **5.** C;

6. $\cos(\pi) = -1$ **7.** $2 + \sum_{k=1}^{\infty} \frac{2x^n}{n}$ the 2 is the constant of integration .

8. (a) $-2 - 6x - 9x^2 - 9x^3 - \cdots - \frac{2 \cdot 3^n \cdot x^n}{n!}$, (b) $f(x) = -2e^{3x}$, (c) below

From the <u>2003 Released Exam</u>

Convergence tests BC 6, 10, 22, 24

Series BC 11, 20, 28, 77

Mathematics and Calculus Related Web Sites:

My Web site: www.LinMcMullin.net and E-mail: lnmcmullin@aol.com

College Board AP Central http://apcentral.collegeboard.com

NCAAPMT Newsletter: Send $5 to Jeff Lucia NCAAPMT Treasurer, 718 Lansdowne Road, Charlotte, NC 28270 (luciaj@pds.charlotte.nc.us). Two issues – late summer, early spring. Or on-line at www.cctt.org/NCAAPMT/ THE BEST $5 YOU'LL SPEND !

Winplot

Winplot http://math.exeter.edu/rparris/default.html Instructions are at http://matcmadison.edu/alehnen/winptut/winpltut.htm Best graphing program around. FREE. Have your students download it and use it too.

General Math Resources

NCTM Homepage http://www.nctm.org/

Math Forum Internet Collection http://mathforum.org/

Calculators and TI

Texas Instruments: and http://education.ti.com/

Calculus in Motion for Geometers Sketchpad www.calculusinmotion.com and for Algebra in Motion

D&S Review Books (Calculus, New York A and B Exams) and ***Teaching AP Calculus*** (2/e) www.dsmarketing.com

On **adapting free-response questions** by Dixie Ross http://apcentral.collegeboard.com/members/article/1,3046,151-165-0-29924,00.html

On **assessment** by Dan Kennedy http://baylor.chattanooga.net/%7Edkennedy/assessment and other stuff http://baylor.chattanooga.net/%7Edkennedy/home