Here's the Graph of the Derivative .... Tell me About the Function.

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Using its derivatives to determine information about the graph of a function is a standard calculus topic and has been for many years. Previously, students worked from the equation of a function, calculated the equation of its derivative and used that to find extreme values, to find where a function was increasing and decreasing, determine concavity, and so forth. The work often ended (20 minutes later) with carefully drawing the graph of the function using all the information that was found. Textbooks used elaborate charts and procedures to be sure all the information was well organized and used them for drawing the graph.

Since the advent of graphing calculator and other technology graphing became a much simpler process: if you want the graph, push some buttons and you'll have it. AP Calculus exams almost never ask students to draw an entire graph. Rather they ask students to determine only certain features of the graph. The same questions may be asked in the context of applications (e.g. find when the particle is farthest to the left = the minimum value of the position function or when is the speed increasing = where do the first and second derivatives have the same sign?). Interpreting graphs may be an entire question or just part of a question.

Furthermore, the student may be asked to work from the graph of the derivative of function without an equation being given. While this should, in theory, make the work easier, students who have never seen this situation have a great deal of difficulty. A question may ask students to use the derivative's graph to determine where the function increases or decreases, where its extreme values are, or to actually find function values by using the area of the region between the graph and the x-axis. Thus, differentiation and integration concepts are required for the same question.

Student difficulties may be blamed partly on textbooks which cover one topic per section, but never put all the varied aspects into one situation or require the student to use ideas from different sections of the textbook together together. Rarely do books work from the graph of the derivative. Therefore, supplementing your textbook is necessary.

Working from the graph of the integrand, the analysis of the graph and other properties of such functions are relatively easy. There are two approaches. One uses derivative ideas, the other integration concepts.

Method 1 Summary

Feature	Conclusion
$y' > 0$	y is increasing
$y' < 0$	y is decreasing
y' changes - to +	y has a local minimum
y' changes + to -	y has a local maximum
y' increasing	y is concave up
y' decreasing	y is concave down
y' extreme values	y has points of inflection

Since "Justify Your Answer" is often part of the question, turning the table around provides the justifications

Conclusion	Justification
y is increasing	$y' > 0$
y is decreasing	$y' < 0$
y has a local minimum	y' changes - to + *
y has a local maximum	y' changes + to - *
y is concave up	y' increasing
y is concave down	y' decreasing
y has points of inflection	y' extreme values

* The first derivative test

Method 2 Summary

Functions are often given in the form $g(x) = g(a) + \int_a^x f(t)\,dt$ often with an initial condition $g(a)$ which may be zero. As *x* increases, the value of the function "accumulates;" the behavior of the function can be seen from the graph of the integrand. It easy to see when the function

- Increases – the graph of the integrand is above the *x*-axis (Riemann sum terms are positive)
- Decreases – the graph of the integrand is below the *x*-axis (Riemann sum terms are negative)
- Attains relative extreme values – the graph of the integrand crosses the *x*-axis (First Derivative Test)
- Is concave up – the graph of the integrand is increasing (Riemann sum terms increase)
- Is concave down – the graph of the integrand is decreasing (Riemann sum terms decrease)
- Has a point of inflection – the graph of the integrand changes from increasing to decreasing or vice versa; i.e. the extreme points of the integrand.

Depending on the complexity of the graph of the integrand it is often possible to determine from the graph (i.e. without finding an antiderivative)

- Function values – by determining geometrically the area of the regions between the graph of the integrand and the *x*-axis
- Derivative values – directly from the graph of the derivative at the *x*-value of interest
- Second derivative value – by determining the slope of the integrand at the *x*-value of interest
- The location of absolute extreme values – by comparing the areas of the regions above and below the *x*-axis along the entire graph.

Students (and certain presenters) often find this method easier than working with derivatives (although both methods should be known and understood). I imagine this is because this method is much more visual than using only the derivative. It is worth spending sometime analyzing graphs from this point of view.

What you can do to prepare your students for this style of question:

First, be very aware that the exam questions cover a range of concepts taught throughout the year. The ideas are all those commonly covered, but by grouping diverse ideas into one question student who have not seen this sort of thing have difficulty. Therefore, this is a topic you must supplement to a great degree.

The concepts may be asked about a generic function or placed in the context of a motion problems or some other "real" situation. Students need to be able to recognize the graphing aspects in terms of these other situations, such as "farthest right" is the "absolute maximum value" and "speed" is the absolute value of velocity which is also is the non-directed distance from the function to the x-axis.

1. Use old AP exam questions and make up similar ones yourself. Use the questions, but discuss what other features can be found and justified in addition to those asked in the question.
2. Find and assign all the questions in your textbook that have a graph in the stem.
3. Almost all exam questions require justification of relative and absolute extreme value (they are different). Practice justifications: be sure to require students to write justifications throughout the year.
4. Use cumulative tests. Since the concepts tested are taught throughout the year, including questions or parts of question using previously taught ideas is a good idea.
5. This is certainly a Rule of 4 issue. Books often use tables or number lines to analyze the derivative. Help students understand what these mean and how the same information looks when presented as a graph or in words.
6. Do not count on your textbook to spiral concepts from previous chapters in their exercises.

What students should know how to do:

Determine information about the function from the derivative, especially where the graph of the derivative and not its equation are given. This may be approached by derivative techniques or antiderivative techniques. These are the kinds of things the questions ask for:

- Find and justify local and absolute extreme values (1^{st} derivative test, 2^{nd} derivative test, closed interval test), from the graph of the derivative (i.e. where the derivative crosses from above the x-axis to below, the function has a relative maximum, etc.)
- Work with functions defined by integrals (accumulation functions)

- Find and justify points of inflection. Where the graph of the derivative has an extreme value, the function has a point of inflection, etc.
- Write an equation of tangent line (read point and slope from graph)
- Evaluate definite integrals (FTC) from graph areas.
- Analyze functions defined by integrals (FTC, accumulation)

Sample AP Questions

The released exams are here:

http://apcentral.collegeboard.com/apc/public/exam/exam_information/index.html

Winplot Demonstrations:

Winplot is free software. PC users google "Winplot" and Mac users google "Winplot for Macs." (There are simple installation instructions for the Mac version.) The program is updated and improved very often; check back about once a month for the latest version. Instruction can be found on the same google pages.

The programs I used can be downloaded from my website http://www.linmcmullin.net/Winplot.html . Look for "Concavity Demo 2." There is an instruction note; Use CTRL+SHIFT+N to toggle it on and off.

AP exam questions:

Free-response:

2011 AB4 1996 AB 1 2009 AB 1 2006 AB 3 2010 AB 5

Multiple-choice

2003 AB – 7, 10, 13, 21, 22, 77, 79, 85, 88 2008 AB – 9, 10, 11, 17, 21, 27, 76, 77, 84, 86

This handout and the PowerPoint slides are at my website http://www.linmcmullin.net/ Click on "AP Calculus"

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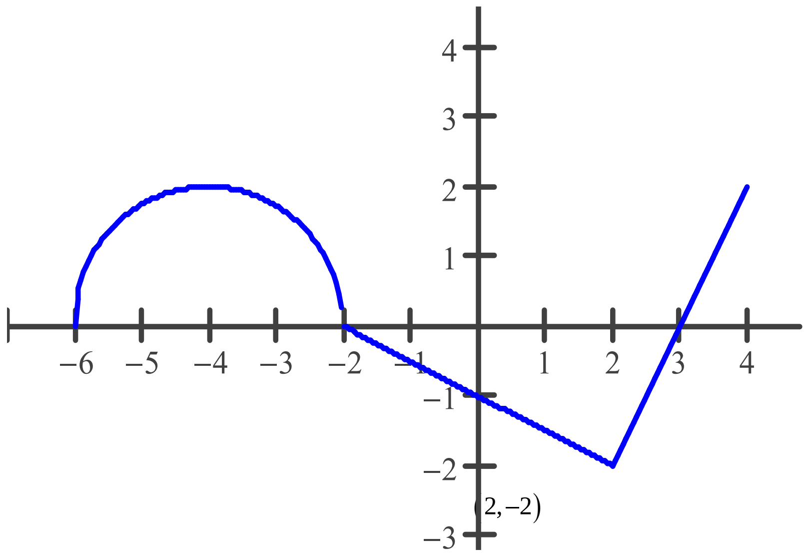
Analysis of AB Calculus Question on Graphing

Year	Given	Extreme Value, inc/dec	f'', POI, and concavity	Tangent line	Values f, f', f'' (integration)	Other
2000 AB3	Graph of f'	Local and absolute	X			
2001 AB4	Equation f'	Local	X	X		
2002 AB4	Graph of f'FTC	Inc/dec	X		X	Sketch graph
2002 AB 4 B	Graph of f'FTC	Absolute	X		X	Trap Rule
2003 AB 4	Graph of f'	Inc/dec	X	X	X - Accumulation	
2003 AB 5 B	Graph of f'FTC		X		X	Average ROC, MVT
2004 AB 5	Graph of f'FTC	Local / absolute	X		X	
2004 AB 4 B	Graph of f'	Local	X	X		
2005 AB 4	Table	Local	X			Sketch graph
2005 AB 4 B	Graph of f'FTC	Inc/dec	X		X	
2006 AB 3	Graph of f'FTC	Local		X	X	Periodic
2006 AB 2 B	Graph of f'	Absolute	X	X	accumulation	
2007 AB6	Equation f'	Local	X			symbolic derivatives
2007 AB4 B	Graph f'	Local / absolute	X			
2008AB4	Equation f'	Absolute	X	X	X	Find limit
2008 AB 5 B	Graph of f'	Absolute	X			Average ROC, MVT
2009 AB4	Graph of f'	Max/min, inc/dec	X			Motion ,IVT
2009 AB 6	Graph of f'	Absolute	X		X	
2009 AB 3 B	Graph of f'FTC		X			Differentiable?, Average ROC, MVT
2009 AB 5 B	Graph of f'	Local	X	X		Composition, Average ROC
2010 AB 5	Graph of f'	Critical points	X		X	Related function
2011 AB 4	Graph of f'	Absolute	X		X	Related function Average ROC, MVT
2011 AB 6 B	Graph of f' FTC	Critical points			X	

BC exams are not listed here. Often the same question appears on both exam; sometimes the BC question also contains BC topics, see for example 2011 BC 6 (Lagrange error bound) or 2011 BC 4 form B (arc length and chain rule).

Group Problem



The graph of a function $f(x)$ for $-6 \le x \le 4$, shown above, consists of a semi-circle and two line segments.

Let $g(x) = 7 + \int_0^x f(t)\,dt$.

This may also be written as $g(x) = g(-6) + \int_{-6}^x f(t)\,dt$ with $g(-6)$ unknown with the initial condition of $g(0) = 7$.

Tell me all the usual stuff about the graph of $g(x)$ *and find the coordinates of the interesting points **without using the word derivative or any concept associated with the derivative.***