Advanced Placement Specialty Conference

TEACHING THE IDEAS BEHIND POWER SERIES

Presented by

LIN McMULLIN

Sequences and Series in Precalculus

Power Series

Intervals of Convergence & Convergence Tests

Error Bounds

Geometric Series

New Series from Old

Problems

Q & A

From the Course Description

*IV. Polynomial Approximations and Series

*** Concept of series. A series is defined as a sequence of partial sums, and convergence is defined in terms of the limit of the sequence of partial sums. Technology can be used to explore convergence or divergence.**

*** Series of constants.**
+ Motivating examples, including decimal expansion.
+ Geometric series with applications.
+ The harmonic series.
+ Alternating series with error bound.
+ Terms of series as areas of rectangles and their relationship to improper integrals, including the integral test and its use in testing the convergence of *p*-series.
+ The ratio test for convergence and divergence.
+ Comparing series to test for convergence or divergence.

*** Taylor series.**

+ Taylor polynomial approximation with graphical demonstration of convergence. (For example, viewing graphs of various Taylor polynomials of the sine function approximating the sine curve.)

+ Maclaurin series and the general Taylor series centered at $x = a$.

+ Maclaurin series for the functions, e^x, sin x, cos x, and $\frac{1}{1-x}$.

+ Formal manipulation of Taylor series and shortcuts to computing Taylor series, including substitution, differentiation, antidifferentiation, and the formation of new series from known series.

+ Functions defined by power series.

+ Radius and interval of convergence of power series.

+ Lagrange error bound for Taylor polynomials.

Precalculus

Sequences and series:

- **Sigma notation,**
- **Recursive and non-recursive definitions of sequences,**
- **Basic formulas for the sums of simple sequences (**$\sum_{k=1}^{n} \text{constant}, \sum_{k=1}^{n} k, \sum_{k=1}^{n} k^2$**, etc.).**
- **Given a sequence they should be able to write the formula for the** n^{th} **term; given the** n^{th} **term they should be able to write the terms of the sequence.**
- **Definition of convergence of a series as the limit of the associated sequence of partial sums.**

Types of Series

- **Arithmetic series,**
- **Geometric series,**
- **Alternating series,**
- **Harmonic series,**
- **Alternating harmonic series,**
- ***p*-series**

Decimals

$$0.\overline{3333}\ldots = 0.3 + 0.03 + 0.003 + 0.003 + \cdots$$

$$= (0.3)10^{0} + (0.3)10^{-1} + (0.3)10^{-2} + (0.3)10^{-3} + \cdots$$

$$= \frac{0.3}{1 - \frac{1}{10}} = \frac{0.3}{0.9} = \frac{1}{3}$$

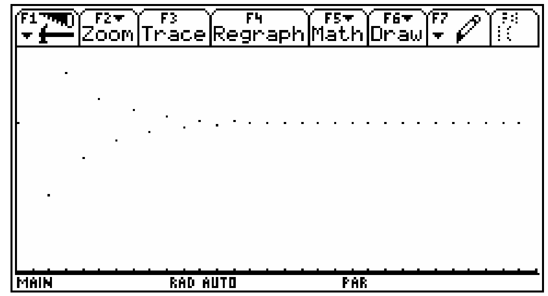
$$0.\overline{999} = (0.9)10^{0} + (0.9)10^{-1} + (0.9)10^{-2} + (0.9)10^{-3} + \cdots$$

$$= \frac{0.9}{1 - \frac{1}{10}} = \frac{0.9}{0.9} = 1$$

$$0.\overline{999}\ldots = 3\left(0.\overline{333}\ldots\right) = 3\left(\frac{1}{3}\right) = 1$$

$$\frac{1 + 0.\overline{999}\ldots}{2} = \frac{1.\overline{999}\ldots}{2} = 0.\overline{999}\ldots = 1$$

To graph $a_n = 1 - (-0.7)^t$ in the plane:



Mode: Parametric

Graph format: Dot

Equation Editor:

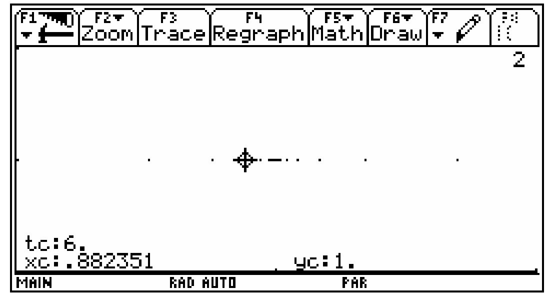
$xt1(t) = t$

$yt1(t) = 1 - (-0.7)$^t

Window:

tmin = 1,	xmin = 0,	ymin = 0,
tmax = 30,	xmax = 31,	ymax = 1.5,
tstep = 1,	xscl = 1,	yscl = 1.

To graph $a_n = 1 - (-0.7)^t$ on a number line:



Mode: Parametric

Graph format: Dot

Equation Editor:

$$xt1(t) = 1 - (-0.7)^\wedge t \quad \text{and} \quad yt1(t) = 1$$

Window:

tmin = 1,	xmin = 0,	ymin = 0,
tmax = 30,	**xmax = 2**,	**ymax = 2**,
tstep = 1,	xscl = 1,	yscl = 1.

Then TRACE the graph and watch it converge.

Taylor Polynomial

If f has n derivatives at c then the n^{th} Taylor polynomial for f at $x = c$ is

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c = 0$, the n^{th} Maclaurin polynomial for f is

$$T_n(x) = f(c) + f'(c)x + \frac{f''(c)}{2!}x^2 + \cdots + \frac{f^{(n)}(c)}{n!}x^n$$

Let $f(x) = x^3 + x^2 - 14x - 24$

a. Write the power series for f centered at $x = 2$.

b. Expand the terms of the power series and simplify.

c. Is this an accident or will this happen with any polynomial? Explain.

$$f(x) = -40 + 2(x-2) + 7(x-2)^2 + (x-2)^3$$

Now expand the binomial terms and see what you get!

1988 BC 4: Determine all values of x for which the series

$\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)}$ **converges.**

$$\lim_{k\to\infty}\left|\frac{\left(\frac{2^{k+1}x^{k+1}}{\ln(k+3)}\right)}{\left(\frac{2^k x^k}{\ln(k+2)}\right)}\right| = \lim_{k\to\infty}|2x|\frac{\ln(k+2)}{\ln(k+3)}$$

By L'Hopital's rule,

$$\lim_{k\to\infty}\frac{\ln(k+2)}{\ln k+3} = \lim_{k\to\infty}\frac{\frac{1}{k+2}}{\frac{1}{k+3}} = \lim_{k\to\infty}\frac{k+3}{k+2} = 1$$

$$\therefore \lim_{k\to\infty}|2x|\frac{\ln(k+2)}{\ln(k+3)} = |2x|$$

$$|2x| < 1 \Leftrightarrow |x| < \frac{1}{2}$$

$\therefore$ Converges for $-\frac{1}{2} < x < \frac{1}{2}$

At $x = \frac{1}{2}$, series becomes $\sum_{k=0}^{\infty} \frac{1}{\ln(k+2)}$

diverges, by comparison with harmonic series $\sum_{k=0}^{\infty} \frac{1}{k+2}$

At $x = -\frac{1}{2}$, series becomes $\sum_{k=0}^{\infty} \frac{(-1)^k}{\ln(k+2)}$

converges, by the alternating series test.

$\therefore$ Series converges for $-\frac{1}{2} \le x < \frac{1}{2}$

Memorize the Maclaurin Series and the interval of convergence for

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{for all } x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad -1 < x < 1$$

2000 BC 3: The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n(n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

$$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$$

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

Ratio test gives $\dfrac{|x-5|}{2} < 1$

$|x-5| < 2$

Radius is 2

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\frac{1}{1000}$.

By the alternating series test the error is less than

$$\left| \frac{(-1)^7\, 7!}{2^7(7+2)} \frac{(6-5)^7}{7!} \right| = \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$$

Is this true at *f*(4)? Why, or why not?

Legrange Form of the Remainder and the Legrange Error Bound

Taylor's Theorem:

If f has derivatives of all orders on an interval containing a, then for any positive integer n and for all x in the interval, <u>there exist a number c between x and a</u> such that:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

This is called the Legrange Form of the Remainder.

Example 1: applying the theorem to the sine function centered at the origin, there exists a *c* between *x* and zero such that

$$\sin x = x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 + \tfrac{1}{6!}\sin(c)x^6$$

$$\sin(0.2) = 0.2 - \tfrac{1}{6}(0.2)^3 - \tfrac{1}{120}(0.2)^5 + \tfrac{1}{720}\sin(c)(0.2)^6$$

$$\sin(0.2) \approx 0.2 - \tfrac{1}{6}(0.2)^3 - \tfrac{1}{120}(0.2)^5 \approx 0.198669\overline{3}$$

Notice that the remainder term is *not* calculated at *x* = 0, but at some *x* = *c* in the interval (0, 0.2), so the sixth power term is used, $R_5 = \tfrac{1}{720}\sin(c)(0.2)^6$ **is not zero. In the open interval (0, 0.2) the largest the sin(*c*) can be is 1: (Note:** $\sin(0.2) \approx 0.19866933...$**), so the largest the error can be is**

$\dfrac{1}{720}(1)(0.2)^6 \approx 8.89\times10^{-8}$ **(or** $\tfrac{1}{720}(0.1986693)(0.2)^6 \approx 1.77\times10^{-8}$**). The actual error is considerably less, about** 2.54×10^{-9}**.**

Example 2: applying the theorem to e^x centered at the origin, there exists a c between x and zero such that

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}e^c x^4$$

Then at, say, $x = 0.2$:

$$e^{0.2} = 1 + (0.2) + \tfrac{1}{2}(0.2)^2 + \tfrac{1}{6}(0.2)^3 + \tfrac{1}{24}e^c(0.2)^4$$

In the interval $[0, 0.2]$ the largest that e^c can be is $e^{0.2} \approx 1.22140$, so the largest the error can be is $\left|\tfrac{1}{24}(1.22140)(0.2)^4\right| \approx 8.1427 \times 10^{-5}$. This is the Lagrange Error Bound. The actual error is about 6.94×10^{-5}.

$$e^{0.2} \approx 1 + 0.2 + \tfrac{1}{2}(0.2)^2 + \tfrac{1}{6}(0.2)^3 = 1.221\overline{333} \pm 8.1427 \times 10^{-5}$$

$$e^{0.2} = 1.22140275816\ldots$$

1999 BC 4: The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$$T_3(f,2) = -3 + 5(x-2) + \tfrac{3}{2}(x-2)^2 - \tfrac{8}{6}(x-2)^3$$
$$f(1.5) \approx -4.958$$

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all x in the closed interval [1.5, 2]. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.

LEB = $\frac{3}{4!}\left|1.5-2\right|^4 =$ **0.0078125,** $f(1.5) > -4.958\overline{3} - 0.0078125 = -4.966 > -5$

(c) Write the fourth degree Taylor polynomial, *P*(*x*), for $g(x) = f(x^2+2)$ **about *x* = 0. Use *P* to explain why *g* must have a relative minimum at x = 0.**

$$T_3(f,2)(x) = -3 + 5(x-2) + \tfrac{3}{2}(x-2)^2 - \tfrac{8}{6}(x-2)^3$$

$$T_2(f,2)(x^2+2) = -3 + 5x^2 + \tfrac{3}{2}x^4$$

and from the coefficients $g'(0) = 0$ and $g''(0) > 0$ **therefore a minimum by the Second Derivative Test.**

Geometric Series

Method 1: **The expression $\frac{1}{1-x}$ is similar to $\frac{a}{1-r}$. If we had a geometric series with a first term of $a = 1$ and a common ratio of x, then $S = \frac{1}{1-x}$.**

Turning this around, the power series for $\frac{1}{1-x}$ must be the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

and the interval of convergence must be all x such that $|x| < 1$ or $-1 < x < 1$.

Rational Expressions

Method 2: **Long division yields the same result:**

$$\begin{array}{r|l} & 1 + x^2 + x^3 + \cdots \\ \hline (1-x) & 1 \\ & \underline{1 - x} \\ & x \\ & \underline{x - x^2} \\ & x^2 \\ & \underline{x^2 - x^3} \\ & \cdots \end{array}$$

Binomial Theorem

Method 3: Expanding $(1-x)^{-1}$ by the Binomial Theorem also gives the same result.

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \text{etc.}$$

$$\begin{aligned}(1-x)^{-1} &= 1^{-1} + (-1)(1)^{-2}(-x)^1 + \frac{(-1)(-2)}{2}(1)^{-3}(-x)^2 \\ &\quad + \frac{(-1)(-2)(-3)}{2\cdot 3}(1)^{-4}(-x)^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots\end{aligned}$$

Rational Expressions

Other functions can be handled in the same ways. One way to find the Maclaurin Series for any rational expression, such as $\frac{15x}{x^2+5}$, is to arrange the terms with the ***lowest power first*** and perform a long division.

$$
\begin{array}{r|l}
 & 3x - \tfrac{3}{5}x^3 + \tfrac{3}{25}x^5 - \tfrac{3}{125}x^7 + \cdots \\
\hline
5 + x^2 & 15x \\
 & \underline{15x + 3x^3} \\
 & -3x^3 \\
 & \underline{-3x^3 - \tfrac{3}{5}x^5} \\
 & \underline{\tfrac{3}{5}x^5 + \tfrac{3}{25}x^7} \\
 & -\tfrac{3}{25}x^7
\end{array}
$$

$$\frac{15x}{5+x^2} = \frac{3x}{1-\left(-\frac{x^2}{5}\right)}$$

This is a geometric series with a first term of 3*x* and a ratio of $r = -\frac{x^2}{5}$

$$\frac{3x}{1-\left(-\frac{x^2}{5}\right)} = 3x + 3x\left(-\frac{x^2}{5}\right) + 3x\left(-\frac{x^2}{5}\right)^2 + 3x\left(-\frac{x^2}{5}\right)^3 + \cdots$$

$$= 3x - \frac{3}{5}x^3 + \frac{3}{25}x^5 - \frac{3}{125}x^7 + \cdots$$

And the interval of convergence is

$$\left|-\frac{x^2}{5}\right| < 1$$

$$|x| < \sqrt{5}$$

$$-\sqrt{5} < x < \sqrt{5}$$

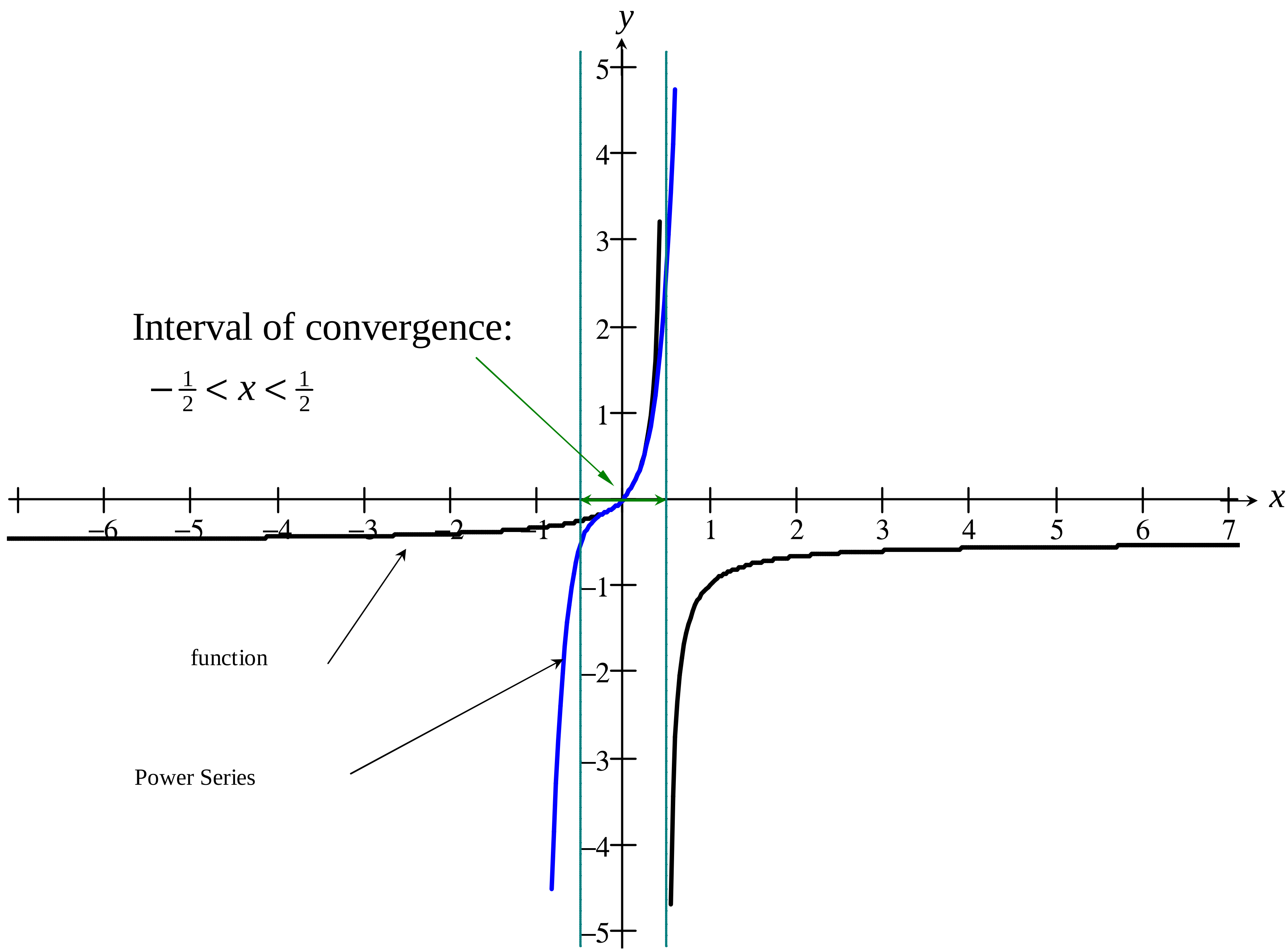
A “Mistake”

$$\frac{x}{1-2x} = x + x(2x) + x(2x)^2 + x(2x)^3 + \cdots$$

$$= x + 2x^2 + 4x^3 + 8x^4 + 16x^5 \cdots + 2^{n-1}x^n + \cdots = \sum_{k=1}^{n} 2^{k-1}x^k$$

The series is geometric and converges when $|2x| < 1$ **or** $-\frac{1}{2} < x < \frac{1}{2}$.

So the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.


Interval of convergence:
$-\frac{1}{2} < x < \frac{1}{2}$
function
Power Series
y
x

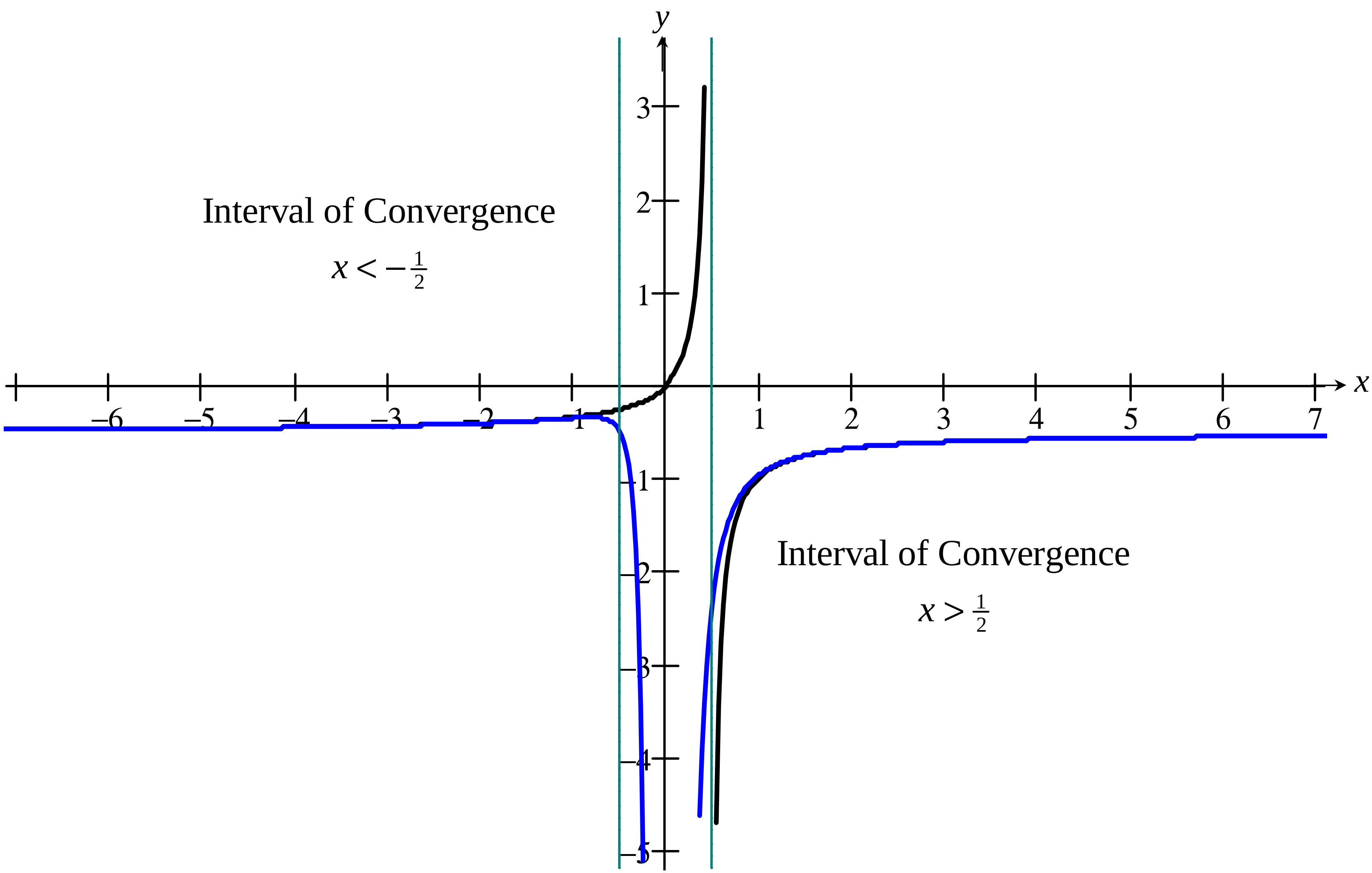
But what if you do this?

$$\frac{x}{1-2x}=\frac{-\frac{1}{2}}{1-\frac{1}{2x}}=-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2x}\right)-\frac{1}{2}\left(\frac{1}{2x}\right)^{2}-\frac{1}{2}\left(\frac{1}{2x}\right)^{3}-\cdots$$

$$=-\frac{1}{2}-\frac{1}{4x}-\frac{1}{8x^{2}}-\frac{1}{16x^{3}}-\frac{1}{32x^{4}}-\cdots-\frac{1}{2^{n}x^{n-1}}-\cdots=\sum_{n=1}^{\infty}\frac{-1}{\left(2^{n}x^{n-1}\right)}$$

The series is geometric and converges when $\left|\frac{1}{2x}\right|<1$ or $\frac{1}{2}<|x|$.

So the interval of convergence is $\left(-\infty,-\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$.


Interval of Convergence
$x < -\frac{1}{2}$
Interval of Convergence
$x > \frac{1}{2}$
y
x
–6
–5
–4
–3
–2
–1
1
2
3
4
5
6
7
3
2
1
–1
–2
–3
–4
–5

New Series from Old

Treating $\frac{1}{1+x^2}$ as a geometric series with $r=-x^2$ gives:

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots=\sum_{k=0}^{n}(-1)^k x^{2k} \quad \text{for } -1<x<1$$

But $\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$. Therefore

$$\tan^{-1}x=\int \frac{1}{1+x^2}dx=C+x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\cdots=C+\sum_{k=1}^{n}\frac{(-1)^{2k-1}}{2k-1}x^{2k-1}$$

The initial condition $\tan^{-1}(0)=0$ tells us that $C = 0$. So the power series for $\tan^{-1}x$ is $\sum_{k=1}^{n}\frac{(-1)^{2k-1}}{2k-1}x^{2k-1}$ for $-1<x<1$.

1993 BC 3

Let f be the function given by $f(x) = e^{\frac{x}{2}}$.

(a) Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$ about $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{x/2} = 1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \cdots + \frac{(x/2)^n}{n!} + \cdots$$

$$e^{x/2} = 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 3!} + \cdots + \frac{x^n}{2^n n!} + \cdots$$

(b) Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about for $x = 0$ for

$$g(x) = \frac{e^{\frac{x}{2}} - 1}{x}.$$

$$e^{x/2} = 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 3!} + \cdots + \frac{x^n}{2^n n!} + \cdots$$

$$\frac{e^{x/2} - 1}{x} = \frac{\left(1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 3!} + \cdots + \frac{x^n}{2^n n!} + \cdots\right) - 1}{x}$$

$$\frac{e^{x/2} - 1}{x} = \frac{1}{2} + \frac{x}{2^2 2!} + \frac{x^2}{2^3 3!} + \cdots + \frac{x^{n-1}}{2^n n!} + \cdots$$

(c) For the function *g* in part (b), find $g'(2)$ and use it to show that

$$\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}.$$

$$g(x) = \frac{1}{2} + \frac{x}{2^2 2!} + \frac{x^2}{2^3 3!} + \cdots + \frac{x^{n-1}}{2^n n!} + \cdots$$

$$g'(x) = \frac{1}{2^2 2!} + \frac{2x}{2^3 3!} + \cdots + \frac{(n-1)x^{n-2}}{2^n n!} + \cdots$$

$$= \frac{1}{8} + \frac{x}{24} + \cdots + \frac{(n-1)x^{n-2}}{2^n n!} + \cdots$$

$$g'(2) = \frac{1}{2^2 2!} + \frac{2 \cdot 2}{2^3 3!} + \cdots + \frac{(n-1)2^{n-2}}{2^n n!} + \cdots$$

$$= \frac{1}{8} + \frac{1}{12} + \cdots + \frac{n-1}{4n!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{n}{4(n+1)!}$$

Also; $g(x) = \dfrac{e^{x/2} - 1}{x}$

$$g'(x) = \frac{x\left(\frac{1}{2}e^{x/2}\right) - \left(e^{x/2} - 1\right)(1)}{x^2}$$

$$g'(2) = \frac{2 \cdot \frac{1}{2}e - (e-1)}{4} = \frac{1}{4}$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$$

Euler's Formula

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

Expand, and simplify the terms. Group those terms without *i* and those with *i*.

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \cdots$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)i$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) i$$

Recognize the Maclaurin Series for $\cos x$ and $\sin x$.

$$e^{ix} = \cos x + i \sin x$$

Finally substitute $x = \pi$ and simplify again.

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$e^{i\pi} = -1 + 0i$$

$$e^{i\pi} + 1 = 0$$