Advanced Placement Specialty Conference

TEACHING THE IDEAS BEHIND POWER SERIES

Presented by
LIN McMULLIN
Sequences and Series in Precalculus

Power Series

Intervals of Convergence & Convergence Tests

Error Bounds

Geometric Series

New Series from Old

Problems

Q & A
*IV. Polynomial Approximations and Series*

* Concept of series. A series is defined as a sequence of partial sums, and convergence is defined in terms of the limit of the sequence of partial sums. Technology can be used to explore convergence or divergence.

* Series of constants.
  + Motivating examples, including decimal expansion.
  + Geometric series with applications.
  + The harmonic series.
  + Alternating series with error bound.
  + Terms of series as areas of rectangles and their relationship to improper integrals, including the integral test and its use in testing the convergence of $p$-series.
  + The ratio test for convergence and divergence.
  + Comparing series to test for convergence or divergence.
* Taylor series.
+ Taylor polynomial approximation with graphical demonstration of convergence. (For example, viewing graphs of various Taylor polynomials of the sine function approximating the sine curve.)
+ Maclaurin series and the general Taylor series centered at $x = a$.
+ Maclaurin series for the functions, $e^x$, $\sin x$, $\cos x$, and $\frac{1}{1-x}$.
+ Formal manipulation of Taylor series and shortcuts to computing Taylor series, including substitution, differentiation, antidifferentiation, and the formation of new series from known series.
+ Functions defined by power series.
+ Radius and interval of convergence of power series.
+ Lagrange error bound for Taylor polynomials.
Precalculus

Sequences and series:

- **Sigma notation,**
- **Recursive and non-recursive definitions of sequences,**
- **Basic formulas for the sums of simple sequences** \((\sum_{k=1}^{n} \text{constant}, \sum_{k=1}^{n} k, \sum_{k=1}^{n} k^2, \text{etc.})\).  
- **Given a sequence they should be able to write the formula for the** \(n^{\text{th}}\) **term; given the** \(n^{\text{th}}\) **term they should be able to write the terms of the sequence.**
- **Definition of convergence of a series as the limit of the associated sequence of partial sums.**
Types of Series

- Arithmetic series,
- Geometric series,
- Alternating series,
- Harmonic series,
- Alternating harmonic series,
- $p$-series
Decimals

\[0.333\ldots = 0.3 + 0.03 + 0.003 + \cdots\]
\[= (0.3)10^0 + (0.3)10^{-1} + (0.3)10^{-2} + (0.3)10^{-3} + \cdots\]
\[= \frac{0.3}{1 - \frac{1}{10}} = \frac{0.3}{0.9} = \frac{1}{3}\]

\[0.999 = (0.9)10^0 + (0.9)10^{-1} + (0.9)10^{-2} + (0.9)10^{-3} + \cdots\]
\[= \frac{0.9}{1 - \frac{1}{10}} = \frac{0.9}{0.9} = 1\]
\[0.999\ldots = 3\left(0.333\ldots\right) = 3\left(\frac{1}{3}\right) = 1\]

\[
\frac{1 + 0.999\ldots}{2} = \frac{1.999\ldots}{2} = 0.999\ldots = 1
\]
To graph \( a_n = 1 - (-0.7)^t \) in the plane:

**Mode:** Parametric

**Graph format:** Dot

**Equation Editor:**

\[
xt1(t) = t
\]

\[
yt1(t) = 1 - (-0.7)^t
\]

**Window:**

\[
t_{\text{min}} = 1, \quad x_{\text{min}} = 0, \quad y_{\text{min}} = 0,
\]

\[
t_{\text{max}} = 30, \quad x_{\text{max}} = 31, \quad y_{\text{max}} = 1.5,
\]

\[
t_{\text{step}} = 1, \quad x_{\text{scl}} = 1, \quad y_{\text{scl}} = 1.
\]
To graph \( a_n = 1 - (-0.7)^t \) on a number line:

**Mode:** Parametric  

**Graph format:** Dot  

**Equation Editor:**

\[
xt1(t) = 1 - (-0.7)^t \quad \text{and} \quad yt1(t) = 1
\]

**Window:**

\[
\begin{align*}
t\text{min} &= 1, & x\text{min} &= 0, & y\text{min} &= 0, \\
t\text{max} &= 30, & x\text{max} &= 2, & y\text{max} &= 2, \\
t\text{step} &= 1, & x\text{sc} &= 1, & y\text{sc} &= 1.
\end{align*}
\]

Then TRACE the graph and watch it converge.
Taylor Polynomial

If \( f \) has \( n \) derivatives at \( c \) then the \( n \)-th Taylor polynomial for \( f \) at \( x = c \) is:

\[
T_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n
\]

If \( c = 0 \), the \( n \)-th Maclaurin polynomial for \( f \) is:

\[
T_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n
\]
Let $f(x) = x^3 + x^2 - 14x - 24$

a. Write the power series for $f$ centered at $x = 2$.

b. Expand the terms of the power series and simplify.

c. Is this an accident or will this happen with any polynomial? Explain.

$$f(x) = -40 + 2(x - 2) + 7(x - 2)^2 + (x - 2)^3$$

Now expand the binomial terms and see what you get!
1988 BC 4: Determine all values of $x$ for which the series

$$
\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)}
$$

converges.

By L'Hopital's rule,

$$
\lim_{k \to \infty} \left| \frac{2^{k+1} x^{k+1}}{\ln(k+3)} \right| = \lim_{k \to \infty} \left| 2x \right| \frac{\ln(k+2)}{\ln(k+3)}
$$

$$
\lim_{k \to \infty} \frac{\ln(k+2)}{\ln(k+3)} = \lim_{k \to \infty} \frac{1}{k+2} = \lim_{k \to \infty} \frac{k+3}{k+3} = 1
$$

$$
\therefore \lim_{k \to \infty} \left| 2x \right| \frac{\ln(k+2)}{\ln(k+3)} = \left| 2x \right|
$$

$$
\left| 2x \right| < 1 \iff \left| x \right| < \frac{1}{2}
$$
Converges for \(-\frac{1}{2} < x < \frac{1}{2}\)

At \(x = \frac{1}{2}\), series becomes \(\sum_{k=0}^{\infty} \frac{1}{\ln(k+2)}\)

diverges, by comparison with harmonic series \(\sum_{k=0}^{\infty} \frac{1}{k+2}\)

At \(x = -\frac{1}{2}\), series becomes \(\sum_{k=0}^{\infty} \frac{(-1)^k}{\ln(k+2)}\)

converges, by the alternating series test.

\[
\therefore \text{ Series converges for } -\frac{1}{2} \leq x < \frac{1}{2}
\]
Memorize the Maclaurin Series and the interval of convergence for

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{for all } x \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x \]

\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad -1 < x < 1 \]
2000 BC 3: The Taylor series about $x = 5$ for a certain function $f$ converges to $f(x)$ for all $x$ in the interval of convergence. The $n$th derivative of $f$ at $x = 5$ is given by $f^{(n)}(5) = \frac{(-1)^n n!}{2^n (n + 2)}$, and $f(5) = \frac{1}{2}$.

(a) Write the third-degree Taylor polynomial for $f$ about $x = 5$.

$$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6} (x - 5) + \frac{1}{16} (x - 5)^2 - \frac{1}{40} (x - 5)^3$$

(b) Find the radius of convergence of the Taylor series for $f$ about $x = 5$.

Ratio test gives $\left|\frac{x - 5}{2}\right| < 1$

$|x - 5| < 2$

Radius is 2
(c) Show that the sixth-degree Taylor polynomial for \( f \) about \( x = 5 \) approximates \( f(6) \) with error less than \( \frac{1}{1000} \).

By the alternating series test the error is less than

\[
\left| \frac{(-1)^7 7! (6 - 5)^7}{2^7 (7 + 2) 7!} \right| = \frac{1}{2^7 (9)} = \frac{1}{1152} < \frac{1}{1000}
\]

Is this true at \( f(4) \)? Why, or why not?
Legrange Form of the Remainder and the Legrange Error Bound

Taylor’s Theorem:
If \( f \) has derivatives of all orders on an interval containing \( a \), then for any positive integer \( n \) and for all \( x \) in the interval, there exist a number \( c \) between \( x \) and \( a \) such that:

\[
f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)
\]

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}
\]

This is called the Legrange Form of the Remainder.
Example 1: applying the theorem to the sine function centered at the origin, there exists a $c$ between $x$ and zero such that

\[
sin x = x - \frac{1}{6} x^3 + \frac{1}{120} x^5 + \frac{1}{6!} \sin(c) x^6
\]

\[
\sin(0.2) = 0.2 - \frac{1}{6} (0.2)^3 - \frac{1}{120} (0.2)^5 + \frac{1}{720} \sin(c)(0.2)^6
\]

\[
\sin(0.2) \approx 0.2 - \frac{1}{6} (0.2)^3 - \frac{1}{120} (0.2)^5 \approx 0.1986693
\]

Notice that the remainder term is not calculated at $x = 0$, but at some $x = c$ in the interval $(0, 0.2)$, so the sixth power term is used, $R_5 = \frac{1}{720} \sin(c)(0.2)^6$ is not zero. In the open interval $(0, 0.2)$ the largest the $\sin(c)$ can be is 1: (Note: $\sin(0.2) \approx 0.19866933...$), so the largest the error can be is

\[
\frac{1}{720} (1)(0.2)^6 \approx 8.89 \times 10^{-8} \text{ (or } \frac{1}{720} (0.1986693)(0.2)^6 \approx 1.77 \times 10^{-8}). \text{ The actual error is considerably less, about } 2.54 \times 10^{-9}.
\]
Example 2: applying the theorem to $e^x$ centered at the origin, there exists a $c$ between $x$ and zero such that

$$e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} e^c x^4$$

Then at, say, $x = 0.2$:

$$e^{0.2} = 1 + (0.2) + \frac{1}{2} (0.2)^2 + \frac{1}{6} (0.2)^3 + \frac{1}{24} e^c (0.2)^4$$

In the interval $[0,0.2]$ the largest that $e^c$ can be is $e^{0.2} \approx 1.22140$, so the largest the error can be is $\left| \frac{1}{24} (1.22140)(0.2)^4 \right| \approx 8.1427 \times 10^{-5}$. This is the Lagrange Error Bound. The actual error is about $6.94 \times 10^{-5}$.

$$e^{0.2} \approx 1 + 0.2 + \frac{1}{2} (0.2)^2 + \frac{1}{6} (0.2)^3 = 1.221333 \pm 8.1427 \times 10^{-5}$$

$$e^{0.2} = 1.22140275816...$$
1999 BC 4: The function $f$ has derivatives of all orders for all real numbers $x$. Assume $f(2) = -3, f'(2) = 5, f''(2) = 3,$ and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for $f$ about $x = 2$ and use it to approximate $f(1.5)$.

$$T_3(f, 2) = -3 + 5(x - 2) + \frac{3}{2} (x - 2)^2 - \frac{8}{6} (x - 2)^3$$

$$f(1.5) \approx -4.958$$

(b) The fourth derivative of $f$ satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all $x$ in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$. 
\[ LEB = \frac{3}{4!} |1.5 - 2|^4 = 0.0078125, \quad f(1.5) > -4.9583 - 0.0078125 = -4.966 > -5 \]

(c) Write the fourth degree Taylor polynomial, \( P(x) \), for \( g(x) = f(x^2 + 2) \) about \( x = 0 \). Use \( P \) to explain why \( g \) must have a relative minimum at \( x = 0 \).

\[
T_3(f, 2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3 \\
T_2(f, 2)(x^2 + 2) = -3 + 5x^2 + \frac{3}{2}x^4
\]

and from the coefficients \( g'(0) = 0 \) and \( g''(0) > 0 \) therefore a minimum by the Second Derivative Test.
**Geometric Series**

*Method 1:* The expression \( \frac{1}{1-x} \) is similar to \( \frac{a}{1-r} \). If we had a geometric series with a first term of \( a = 1 \) and a common ratio of \( x \), then \( S = \frac{1}{1-x} \).

Turning this around, the power series for \( \frac{1}{1-x} \) must be the geometric series

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots
\]

and the interval of convergence must be all \( x \) such that \( |x| < 1 \) or \(-1 < x < 1\).
Rational Expressions

Method 2: Long division yields the same result:

\[
\frac{1 + x^2 + x^3 + \cdots}{(1 - x)1} = \frac{1}{1 - x} - \frac{x}{x - x^2} + \frac{x^2}{x^2 - x^3} - \cdots
\]
 BINOMIAL THEOREM

Method 3: Expanding $(1 - x)^{-1}$ by the Binomial Theorem also gives the same result.

\[
(a + b)^n = \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \text{etc.}
\]

\[
(1 - x)^{-1} = 1 + (1)(1)(-x)^1 + \frac{(-1)(2)}{2!} (-x)^2 + \frac{(-1)(2)(-3)}{3!} (-x)^3 + \ldots
\]

\[
= 1 + x + x^2 - x^3 + \ldots
\]
Rational Expressions

Other functions can be handled in the same ways. One way to find the Maclaurin Series for any rational expression, such as \( \frac{15x}{x^2 + 5} \), is to arrange the terms with the \textit{lowest power first} and perform a long division.

\[
\begin{array}{c}
\phantom{5 + x^2)} & 3x - \frac{3}{5} x^3 + \frac{3}{25} x^5 - \frac{3}{125} x^7 + \cdots \\
5 + x^2) & 15x \\
\phantom{5 + x^2)} & 15x + 3x^3 \\
\phantom{5 + x^2)} & -3x^3 \\
\phantom{5 + x^2)} & -3x^3 - \frac{3}{5} x^5 \\
\phantom{5 + x^2)} & \frac{3}{5} x^5 + \frac{3}{25} x^7 \\
\phantom{5 + x^2)} & -\frac{3}{25} x^7
\end{array}
\]
\[
\frac{15x}{5 + x^2} = \frac{3x}{1 - \left(-\frac{x^2}{5}\right)}
\]

This is a geometric series with a first term of \(3x\) and a ratio of \(r = -\frac{x^2}{5}\)

\[
\frac{3x}{1 - \left(-\frac{x^2}{5}\right)} = 3x + 3x\left(-\frac{x^2}{5}\right) + 3x\left(-\frac{x^2}{5}\right)^2 + 3x\left(-\frac{x^2}{5}\right)^3 + \cdots
\]

\[
= 3x - \frac{3}{5}x^3 + \frac{3}{25}x^5 - \frac{3}{125}x^7 + \cdots
\]

And the interval of convergence is

\[
\left| -\frac{x^2}{5} \right| < 1
\]

\[
|x| < \sqrt{5}
\]

\[
-\sqrt{5} < x < \sqrt{5}
\]
A “Mistake”

\[ \frac{x}{1-2x} = x + x(2x) + x(2x)^2 + x(2x)^3 + \cdots = x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + \cdots = \sum_{k=1}^{n} 2^{k-1} x^k \]

The series is geometric and converges when \( |2x| < 1 \text{ or } -\frac{1}{2} < x < \frac{1}{2} \).

So the interval of convergence is \( -\frac{1}{2} < x < \frac{1}{2} \).
Interval of convergence:

\[-\frac{1}{2} < x < \frac{1}{2}\]

Power Series
But what if you do this?

\[
\frac{x}{1-2x} = \frac{-\frac{1}{2}}{1-\frac{1}{2x}} = -\frac{1}{2} - \frac{1}{2} \left( \frac{1}{2x} \right) - \frac{1}{2} \left( \frac{1}{2x} \right)^2 - \frac{1}{2} \left( \frac{1}{2x} \right)^3 - \ldots
\]

\[
= -\frac{1}{2} - \frac{1}{4x} - \frac{1}{8x^2} - \frac{1}{16x^3} - \frac{1}{32x^4} \ldots - \frac{1}{2^n x^{n-1}} \ldots = \sum_{n=1}^{\infty} \frac{-1}{2^n x^{n-1}}
\]

The series is geometric and converges when \( \left| \frac{1}{2x} \right| < 1 \) or \( \frac{1}{2} < |x| \).

So the interval of convergence is \((-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)\).
Interval of Convergence

$x < -\frac{1}{2}$

Interval of Convergence

$x > \frac{1}{2}$
New Series from Old

Treating $\frac{1}{1+x^2}$ as a geometric series with $r = -x^2$ gives:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{k=0}^{n} (-1)^k x^{2k} \text{ for } -1 < x < 1$$

But $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$. Therefore

$$\tan^{-1} x = \int \frac{1}{1+x^2} \, dx = C + x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \cdots = C + \sum_{k=1}^{n} \frac{(-1)^{2k-1}}{2k-1} x^{2k-1}$$

The initial condition $\tan^{-1}(0) = 0$ tells us that $C = 0$. So the power series for $\tan^{-1} x$ is

$$\sum_{k=1}^{n} \frac{(-1)^{2k-1}}{2k-1} x^{2k-1} \text{ for } -1 < x < 1.$$
Let $f$ be the function given by $f(x) = e^x$.

(a) Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$ about $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x/2} = 1 + \frac{x}{2} + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \cdots$$

$$e^{ix/2} = 1 + \frac{ix}{2} + \frac{(ix/2)^2}{2!} + \frac{(ix/2)^3}{3!} + \cdots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$
(b) Use the result from part (a) to write the first three nonzero terms and
the general term of the series expansion about \( x = 0 \) for

\[
g(x) = \frac{e^{x/2} - 1}{x}.
\]

\[
e^{x/2} = 1 + \frac{x}{2} + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \cdots + \frac{x^n}{n \cdot n!} + \cdots
\]

\[
\frac{e^{x/2} - 1}{x} = \left(1 + \frac{x}{2} + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \cdots + \frac{x^n}{n \cdot n!} + \cdots\right) - 1
\]

\[
= \frac{1}{2} + \frac{x}{2 \cdot 2!} + \frac{x^2}{3 \cdot 3!} + \cdots + \frac{x^{n-1}}{n \cdot n!} + \cdots
\]
(c) For the function $g$ in part (b), find $g'(2)$ and use it to show that

$$\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}.$$

$$g(x) = \frac{1}{2} + \frac{x}{2^22!} + \frac{x^2}{2^33!} + \cdots + \frac{x^{n-1}}{2^n n!} + \cdots$$

$$g'(x) = \frac{1}{2^22!} + \frac{2x}{2^33!} + \cdots + \frac{(n-1)x^{n-2}}{2^n n!} + \cdots$$

$$= \frac{1}{8} + \frac{x}{24} + \cdots + \frac{(n-1)x^{n-2}}{2^n n!} + \cdots$$

$$g'(2) = \frac{1}{2^22!} + \frac{2}{2^33!} + \cdots + \frac{(n-1)2^{n-2}}{2^n n!} + \cdots$$

$$= \frac{1}{8} + \frac{1}{12} + \cdots + \frac{n-1}{4n!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{n}{4(n+1)!}$$

Also; 

$$g(x) = \frac{e^{x/2} - 1}{x}$$

$$g'(x) = \frac{x \left( \frac{1}{2} e^{x/2} \right) - (e^{x/2} - 1)}{x^2}$$

$$g'(2) = \frac{2 \cdot \frac{1}{2} e - (e - 1)}{4} = \frac{1}{4}$$

$$\therefore \sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$$
Euler’s Formula

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \]

\[ e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \]

Expand, and simplify the terms. Group those terms without \( i \) and those with \( i \).

\[ e^{ix} = 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \frac{i^5 x^5}{5!} + \cdots \]

\[ e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)i \]
\[ e^{ix} = \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) + \left( \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \right)i \]

Recognize the Maclaurin Series for \( \cos x \) and \( \sin x \).

\[ e^{ix} = \cos x + i\sin x \]

Finally substitute \( x = \pi \) and simplify again.

\[ e^{i\pi} = \cos \pi + i\sin \pi \]

\[ e^{i\pi} = -1 + 0i \]

\[ e^{i\pi} + 1 = 0 \]