Painting a Sphere
From *Teaching AP Calculus* by Lin McMullin

Suppose you started with a point, the origin to be specific, and painted it. You put on layer and layers of paint until your point grows to a sphere with radius *r*. Let's stop and admire your work part way through the job; at this point the radius is x_i and $0 \le x_i \le r$.

How much paint will you need for the next layer?

Easy: you need an amount equal to the surface area of the sphere, $4\pi x_i^{\,2}$, times the thickness of the paint. As everyone knows by now the paint is thin, specifically Δx thin. So we add an amount of paint to the sphere of $4\pi x^2 \left(\Delta x\right)$. The volume of the final sphere must be the same as the total amount of paint. The total amount of paint must be the (Riemann) sum of all the layers or $\sum_{i=1}^{n} 4\pi x_i^{\,2}\left(\Delta x\right)$. As usual Δx is very thin, tending to zero as a matter of fact, so the amount of paint must be

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} 4\pi x_i^{\,2}\left(\Delta x\right) = \int_0^r 4\pi x^2 dx = \tfrac{4}{3}\pi x^3\Big|_0^r = \tfrac{4}{3}\pi r^3 .$$

ALL definite integrals have a Riemann Sum lurking around somewhere. A standard related rate problem (Chapter 10) is to show that the rate of change of the volume of a sphere is proportional to its surface area – the constant of proportionality is *dr/dt*. So it should not be a surprise that the integral of this rate of change is the volume. Interestingly this approach works other places as long as you properly define "radius:"

- A circle centered at the origin with radius *x* and perimeter of $2\pi x$, gains area at a rate equal to its perimeter times the "thickness of the edge" *dx:*

  $$A = \lim_{n \to \infty} \sum_{k=1}^{n} 2\pi x \Delta x = \int_0^r 2\pi x\, dx = \pi r^2$$

- A square centered at the origin with "radius" *x* and sides of length 2x, gains area at a rate equal to its perimeter (8*x*) times the "thickness of the edge" *dx.*

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} 8x \Delta x = \int_0^s 8x \, dx = 4x^2 = (2x)^2 = s^2$$

- A cube centered at the origin with “radius” *x* and edges of length 2*x*, gains volume at a rate equal to its surface area, $6(4x^2)$ times the “thickness of the face” *dx*.

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} 6(4\mathrm{x}^2) \Delta x = \int_0^s 24x^2 dx = 8x^3 = (2x)^3 = s^3$$