Teaching Limits so that Students will Understand Limits

Presented by

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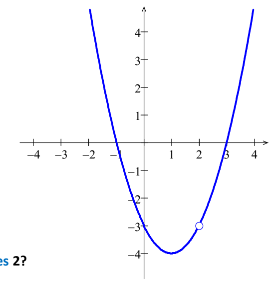
National Math and Science Initiative

Continuity

$$f(x)=\frac{x^3-4x^2+x+6}{x-2}$$



What happens at *x* = 2?

What is *f*(2)?

What happens near *x* = 2?

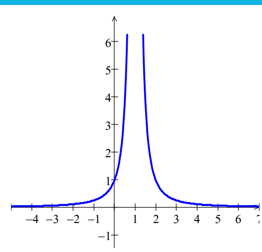
$f(x)$ is near −3

What happens as *x* approaches 2?

$f(x)$ approaches −3

Asymptotes

$$g(x)=\frac{1}{(x-1)^2}$$



What happens at *x* = 1?

What happens near *x* = 1?

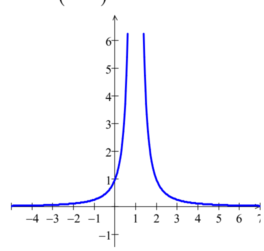
As *x* approaches 1, *g* increases without bound, or *g* approaches infinity.

As *x* increases without bound, *g* approaches 0.

As *x* approaches infinity *g* approaches 0.

Asymptotes

$$g(x)=\frac{1}{(x-1)^2}$$



x	x-1	1/(x-1)^2
0.9	-0.1	100.00
0.91	-0.09	123.46
0.92	-0.08	156.25
0.93	-0.07	204.08
0.94	-0.06	277.78
0.95	-0.05	400.00
0.96	-0.04	625.00
0.97	-0.03	1,111.11
0.98	-0.02	2,500.00
0.99	-0.01	10,000.00
1	0	Undefined
1.01	0.01	10,000.00
1.02	0.02	2,500.00
1.03	0.03	1,111.11
1.04	0.04	625.00
1.05	0.05	400.00
1.06	0.06	277.78
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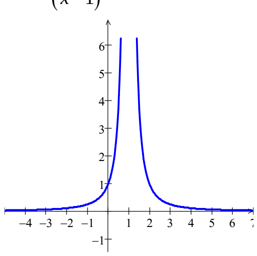
1.08	0.08	156.25
1.09	0.09	123.46
1.10	0.1	100.00

Asymptotes

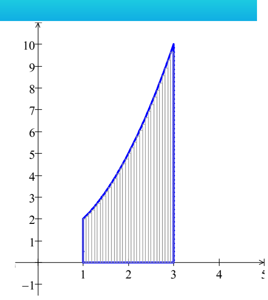
$$g(x)=\frac{1}{(x-1)^2}$$



x	x-1	1/(x-1)^2
1	0	Undefinned
2	1	1
5	4	0.25
10	9	0.01234567901234570
50	49	0.00041649312786339
100	99	0.00010203040506071
500	499	0.00000401604812832
1,000	999	0.00000100200300401
10,000	9999	0.00000001000200030
100,000	99999	0.00000000010000200
1,000,000	999999	0.00000000000100000
10,000,000	9999999	0.00000000000001000
100,000,000	99999999	0.00000000000000010

The Area Problem

$$h(x)=1+x^2$$
$$j(x)=0$$
$$x=1$$
$$x=3$$

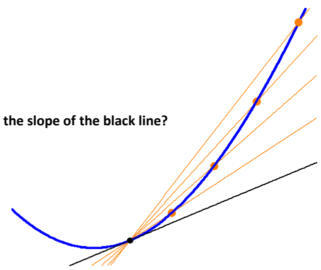


What is the area of the outlined region?

As the number of rectangles increases with out bound, the area of the rectangles approaches the area of the region.

The Tangent Line Problem

What is the slope of the black line?



As the red point approaches the black point, the red secant line approaches the black tangent line, and

The slope of the secant line approaches the slope of the tangent line.

As *x* approaches 1, (5 – 2*x*) approaches ?

x	$5-2x$
0.90	3.20
0.91	3.18
0.92	3.16
0.93	3.14
0.94	3.12
0.95	3.10
0.96	3.08
0.97	3.06
0.98	3.04
0.99	3.02
1.00	**3.00**
1.01	2.98
1.02	2.96
1.03	2.94
1.04	2.92
1.05	2.90
1.06	2.88
1.07	2.86
1.08	2.84
1.09	2.82

x within 0.08 units of 1

x within 0.04 units of 1

f(*x*) within 0.08 units of 3

f(*x*) within 0.16 units of 3

$\lim_{x\to 1}(5-2x)=3$

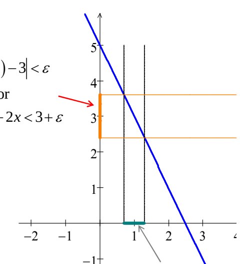
$$|x-1|<\frac{\varepsilon}{2}$$
$$|2x-2|<\varepsilon$$
$$|2-2x|<\varepsilon$$
$$|5-2x-3|<\varepsilon$$
$$|(5-2x)-3|<\varepsilon$$
$$|f(x)-L|<\varepsilon$$

$\lim_{x\to 1}(5-2x)=3$

$|(5-2x)-3|<\varepsilon$

or

$3-\varepsilon<5-2x<3+\varepsilon$

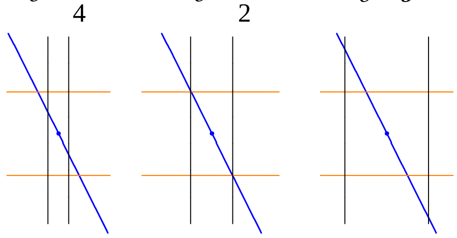


Graph

$|x-1|<\frac{\varepsilon}{2}$ or $1-\frac{\varepsilon}{2}<x<1+\frac{\varepsilon}{2}$

$\lim_{x\to 1}(5-2x)=3$

$\delta=\frac{\varepsilon}{4}$ $\quad$ $\delta=\frac{\varepsilon}{2}$ $\quad$ $\delta=\varepsilon$



The Definition of Limit at a Point

When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it as little as one wishes, this last is called the limit of all the others.

Augustin-Louis Cauchy (1789 – 1857)

The Definition of Limit at a Point

$\lim_{x \to a} f(x) = L$ if, and only if, for any number $\varepsilon > 0$

there is a number $\delta(\varepsilon) > 0$ such that

if $0 < |x - a| < \delta(\varepsilon)$, then $|f(x) - L| < \varepsilon$

Karl Weierstrass (1815 – 1897)

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$\lim_{x \to a} f(x) = L$ if, and only if, for any number $\varepsilon > 0$

there is a number $\delta(\varepsilon) > 0$ and $x \neq a$ such that

if $a - \delta(\varepsilon) < x < a + \delta(\varepsilon)$, then $L - \varepsilon < f(x) < L + \varepsilon$

Karl Weierstrass (1815 – 1897)

Footnote: The Definition of Limit at a Point

$\lim_{x \to a} f(x) = L \Leftrightarrow \forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that

$|f(x) - L| < \varepsilon$, whenever $0 < |x - a| < \delta$

Footnote: The Definition of Limit at a Point

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$\lim_{x \to 1}(5 - 2x) = 3$

x	$5-2x$
0.90	3.20
0.91	3.18
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x within 0.08 units of 1

x within 0.04 units of 1

f(x) within 0.08 units of 3

f(x) within 0.16 units of 3

$\lim_{x \to 1}(5 - 2x) = 3$

$$|f(x) - L| < \varepsilon$$
$$|(5 - 2x) - 3| < \varepsilon$$
$$|5 - 2x - 3| < \varepsilon$$
$$|2 - 2x| < \varepsilon$$
$$|2x - 2| < \varepsilon$$
$$|x - 1| < \frac{\varepsilon}{2}$$

$\lim_{x \to 3} x^2 = 9$

$$|x^2 - 9| < \varepsilon$$
$$|x+3||x-3| < \varepsilon$$

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Near $x = 3$, specifically in $(2,4)$, $5 < |x+3| < 7$

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$$|x^2 - 9| < \varepsilon$$
$$|x+3||x-3| < \varepsilon$$

Near $x = 3$, specifically in $(2,4)$, $5 < |x+3| < 7$

$$7|x-3| < \varepsilon$$
$$|x-3| < \frac{\varepsilon}{7}$$

$\lim_{x \to 3} x^2 = 9$

$$|x^2 - 9| < \varepsilon$$
$$|x+3||x-3| < \varepsilon$$

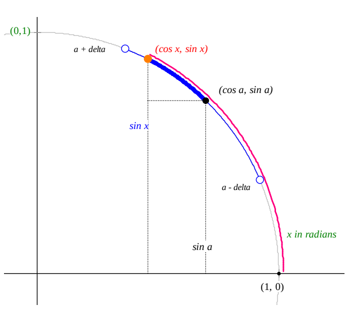
Near $x = 3$, specifically in $(2,4)$, $5 < |x+3| < 7$

$$7|x-3| < \varepsilon$$
$$|x-3| < \frac{\varepsilon}{7}$$

$\delta(\varepsilon) =$ the smaller of 1 and $\frac{\varepsilon}{7}$

Graph

$\lim_{x \to a} \sin(x) = \sin(a)$



Graph

One-sided Limits

$\lim_{x \to a^+} f(x) = L$ if, and only if, for any number $\varepsilon > 0$
there is a number $\delta(\varepsilon) > 0$ such that
if $0 < x - a < \delta(\varepsilon)$, then $|f(x) - L| < \varepsilon$

$\lim_{x \to a^-} f(x) = L$ if, and only if, for any number $\varepsilon > 0$
there is a number $\delta(\varepsilon) > 0$ such that
if $0 < a - x < \delta(\varepsilon)$, then $|f(x) - L| < \varepsilon$

Limits Equal to Infinity

$\lim_{x \to a} f(x) = \infty$ if, and only if, for any number $M > 0$

there is a number $\delta(\varepsilon)$ such that

if $0 < |x-a| < \delta(\varepsilon)$, then $f(x) > M$ $\quad (M < f(x) < \infty)$

Graphically this is a vertical asymptote

$\lim_{x \to a} f(x) = -\infty$ if, and only if, for any number $M < 0$

there is a number $\delta(\varepsilon)$ such that

if $0 < |x-a| < \delta(\varepsilon)$, then $f(x) < M$

Limit as x Approaches Infinity

$\lim_{x \to \infty} f(x) = L$ if, and only if, for any number $\varepsilon > 0$

there is a number $M > 0$ such that

if $x > M$, then $|f(x) - L| < \varepsilon$

Graphically, this is a horizontal asymptote

$\lim_{x \to -\infty} f(x) = L$ if, and only if, for any number $\varepsilon > 0$

there is a number $M < 0$ such that

if $x < M$, then $|f(x) - L| < \varepsilon$

Limit Theorems

Almost all limit are actually found by substituting the values into the expression, simplifying, and coming up with a number, the limit.

The theorems on limits of sums, products, powers, etc. justify the substituting.

Those that don't simplify can often be found with more advanced theorems such as L'Hôpital's Rule

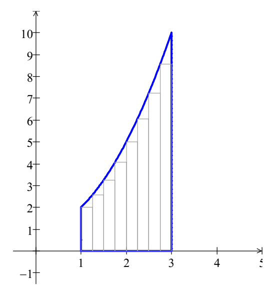
The Area Problem

$h(x) = 1 + x^2$

$j(x) = 0$

$x = 1$

$x = 3$



The Area Problem

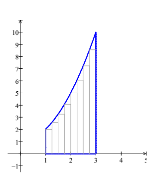
length $= h(x) - j(x) = 1 + x^2$

width $= \dfrac{b-a}{n} = \dfrac{3-1}{n} = \dfrac{2}{n}$

x-coordinates $= \left\{ 1, 1+\tfrac{2}{n}, 1+\tfrac{2}{n}(2), 1+\tfrac{2}{n}(3), \ldots, 1+\tfrac{2}{n}(n) \right\}$

Area $\approx \sum_{i=1}^{n} \left(1 + \left(1 + \tfrac{2}{n} i\right)^2\right)\left(\tfrac{2}{n}\right)$

Area $= \lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + \left(1 + \tfrac{2}{n} i\right)^2\right)\left(\tfrac{2}{n}\right) = \dfrac{32}{3}$



The Area Problem

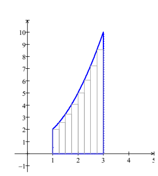
$$\lim_{i \to \infty} \sum_{i=1}^{n} \left(1 + \left(1 + \tfrac{2}{n} i\right)^2\right)\left(\tfrac{2}{n}\right) = \lim_{n \to \infty} \tfrac{2}{n} \sum_{i=1}^{n} 2 + \lim_{n \to \infty} \tfrac{2}{n} \sum_{i=1}^{n} \tfrac{4}{n} i + \lim_{n \to \infty} \tfrac{2}{n} \sum_{i=1}^{n} \tfrac{4}{n^2} i^2$$

$$= \lim_{n \to \infty} \tfrac{4}{n} \sum_{i=1}^{n} 1 + \lim_{n \to \infty} \tfrac{8}{n^2} \sum_{i=1}^{n} i + \lim_{n \to \infty} \tfrac{8}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n \to \infty} \tfrac{4}{n} n + \lim_{n \to \infty} \tfrac{8}{n^2} \frac{n(n+1)}{2} + \lim_{n \to \infty} \tfrac{8}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= 4 + 4 + \tfrac{8}{3}$$

$$= \tfrac{32}{3}$$



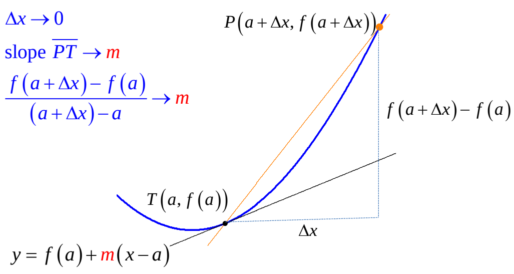
The Tangent Line Problem

As $P \to T$

$\Delta x \to 0$

slope $\overline{PT} \to m$

$$\frac{f(a+\Delta x) - f(a)}{(a+\Delta x) - a} \to m$$



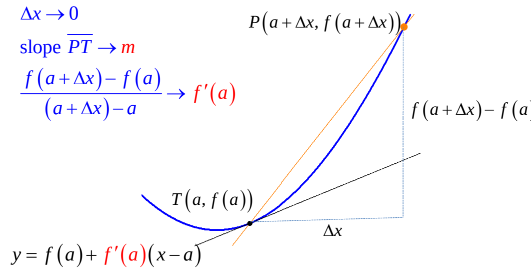
The Tangent Line Problem

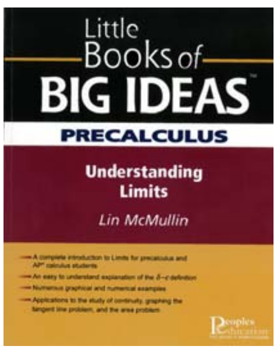
As $P \to T$

$\Delta x \to 0$

slope $\overline{PT} \to m$

$$\frac{f(a+\Delta x) - f(a)}{(a+\Delta x) - a} \to f'(a)$$





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