Discussion of the 2011 AP Calculus free-response questions (operational forms)

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This is a discussion of the questions on the operational forms of the 2011 AP Calculus AB and BC exams. The purpose is to make the reader aware of the approach and style of the questions – what topics the exams test and how they test them – and in hopes that teachers will adapt their approach as suggested here.

In order to do this analysis I consulted three popular textbooks which shall remain nameless. The reason is because I am not trying to criticize any of the textbooks, but only show how the organization of the exam questions and the organization of any good textbook differ. I did this by perusing the index, table of contents and paging thru the sections. I may have missed things, especially if they appeared only in the exercises, but this does not affect the conclusions.

The table at the end of this article shows the results of the analysis.

- The heading for each section give the question number, the general type of question and the way the information given in the stem of each question.
  Notice: The "Rule of Four" is well represented with equations, graphs, tables and questions requiring reading and interpretation all in evidence. A look from year to year will show that the same type question, such as the motion question, can be and has been asked from an analytic, graphical or table stem. (Compare 2011 AB 1, 20008 AB4/BC4, 2006 AB4)
- The topics tested in each of the nine questions are listed in the left column.
- In the next three columns are the chapter and section numbers the question topics appear in each textbook. Parentheses indicate that the topic is in the section cited, but not close to the way the topic appears in the exam question. For example, question AB 1 part (d) asks for when a moving particle changes direction, This occurs at the local maximum or minimum of the function. So this is a max/min question and that is what you find in most textbooks, but not in the context of a moving particle.
  Notice: Most questions have parts that come from different *chapters* of the textbooks (not sections in the same chapter). Usually, the area/volume question has parts that come from 3 contiguous sections in the same chapter (area between curves, volume of a solid with regular cross section and volume of rotation); this year even one part of this question came from a much earlier chapter. The exception here is that the parametric equation and the Taylor series question on the BC exams have topics covered in the same chapter.
  Notice: There are some questions that are not covered in any of the textbooks. These are most often questions asking students to "Justify your answer", or to explain the meaning of a definite integral in the context of the question, or other writing prompts. While some, not all, textbooks do contain writing questions they are not often of the type required on the exams. Another topic is functions defined by integrals.

Conclusions for teachers

This is not intended to be a criticism of the exams for not following the textbooks. Textbooks typically have exercises that reinforce the concepts in the section they follow. For the most part, they make little or no attempt to combine diverse topics into one question or series of questions that includes important ideas from previous sections, let alone previous chapters. It is the textbooks that need changing; the exam questions, at least in my opinion, are universally excellent precisely because they do what the textbooks do not.

Another concern about textbooks is that they often do not show the graphic and numeric considerations of important concepts. Proving theorems is given place well above explaining what the theorems mean graphically and numerically. For example, 2011 AB5/BC5 part (b) asked whether the tangent line approximation (a number) found in part (a) was an over or under estimate. The solution is based on the concavity (a graphical feature) which students were to deduce from the second derivative that they were asked to find analytically.

All this implies (I hope) some things that AP Calculus teachers should do all year.

- Use the "Rule of Four." For every concept find ways to show how it work or what it does numerically, analytically and graphically. Understanding the concept in all these ways is more important than being able to prove the theorem. Often multiple-choice questions from released exams or a good review book will show you ways to ask the same question graphically or numerically; use them throughout the year.

- Make your students read mathematics. Questions AB 2/BC2, and AB 5/BC 5 require students to deal with concepts in context. Stress things like the fact that "rate", "derivative", "slope", and "instantaneous rate of change" all mean the same thing and are closely related to "average rate of change." Likewise, "farthest right" means "absolute maximum" etc.

- Make your students write mathematics. "Justify your answer", "Explain your reasoning", and explain what a definite integral or a derivate at a point means are very common AP exam questions. The exam may also ask why a particular theorem applies or does not apply to the function under consideration. Students will not often be able to do these things unless you have taught them to.

- Since the textbooks don't, you need to combine diverse ideas and concepts into single questions. Use released AP calculus exam questions all year long. As the year goes on find parts of free-response that you can use.

- Spend at least two weeks at the end of the year teaching each type of exam question using past exams. Use several question of each type over a few days. Emphasize how the diverse concepts can fit into the same situation. I used to think of this as part of reviewing for the exam; now I look at it as analyzing, applying, and evaluating what they have learned and using it to create good answers as a main part of what should be taught in the course.

Finally, lest I be accused of teaching-to-the-exam, let me point out that all these things I suggest are good mathematics and good pedagogy. The AP exams are excellent exams precisely because they require students to do the things in ways textbooks do not. But students need to see and work with all this before the exams.

2011 Calculus Operational Exams Analysis

Question	Textbook 1	Textbook 2	Textbook 3
AB 1 Particle Motion Stem: analytic			
(a) Speed	3.4	(2.2)	3.4
(b) Average Velocity	2.1	(2.2)	3.4
(c) Total Distance	5.5	-	7.1
(d) Max/min (1st D test)	(4.2), (4.7)	(3.3)	(4.4), 4.5
AB 2 / BC 2 Stem: Words and Table			
(a) Approximate rate (*y*')	3.1	(P1)	2.4, 3.1
(b) Explain Integral	-	-	-
Trap Approximation*	(5.1), (7.8)	(4.6)	(5.5)
Average value	6.2	4.4	5.3
(c) FTC (evaluate)	5.3	4.4	5.4
Explain integral	-	-	-
(d) Accumulation**	5.5	(4.4)	(5.4), (7.1)
AB 3 Area / Volume Stem: analytic with graph			
(a) Eq of Tangent line	3.1	(2.1)(3.9)	2.4
(b) Area	6.1	7.1	7.2
(c) Volume rotate	6.3	7.2	7.3
AB 4 / BC 4 The graph of the derivative Stem: Graph			
(a) *f* defined by integral	-	-	5.4
FTC	5.3	4.4	5.4
Evaluate from graph	-	4.4	5.4
(b) max/min	4.3, 4.7	(3.6)	4.3
Justify	-	-	-
(c) POI	4.4	3.4	4.3
Give reason	-	-	-
(d) Average ROC	2.1	(P1)	2.4
MVT	4.3	3.2	4.2
AB 5 / AB 5 Differential Equation Stem: Words and Analytic			
(a) Tangent line approx.	4.1	3.9	4.5
(b) find *y*'' (Implicit differentiation)	3.10	2.5	3.7
Over/under estimate	-	-	-
(c) IVP	4.9, 9.1	6.3	6.4
AB 6 Analyze Function Stem: Piecewise analytic			
(a) Continuity	2.4	1.4	2.3
(b) Find *f* ' (Chain rule)	3.7	2.4	3.6
Evaluate *f* '	3.5	2.4	3.6
(c) Average Value	6.2	4.4	5.3

Question	Textbook 1	Textbook 2	Textbook 3
BC 1 Particle Motion in a Plane Stem: analytic			
(a) Speed	11.2	(12.3)	10.2
Acceleration vector	(3.5)	(12.3)	10.2
(b) Slope of Tangent	11.7	(12.3)	-
(c) Position (Accumulation**)	(5.5)	(4.4)	10.2
(d) total distance	(11.2)	-	10.2
BC 3 Area / volume Stem: Analytic with graph			
(a) Arc length	(11.2)	7.4	(10.1)
(b) Volume rotate	6.3	7.2	7.3
(c) Related Rate	3.11	2.6	4.6
BC 6 Power Series Stem; Analytic (with graph for (d))			
(a) Taylor for sin (*x*)	8.4	9.7	9.2
Substitute for x^2	10.7	9.10	-
(b) Taylor for cos (*x*)	8.4	9.7	9.2
(c) Find coefficient	(8.4)	(9.7)	9.2
(d) Lagrange Error Bound	8.4***	9.7	9.3

* All books cover the Trapezoidal *Rule* but none do a trapezoid *approximation* where use of the formula may not be applicable and certainly isn't necessary or expected.

** By which I mean use of the equation $f(x) = f(a) + \int_a^x f'(t)\,dt$ which has been heavily used in past years, but not so much this year. One of the most useful equations in calculus, it is not mentioned in any textbook.

** The formula is in text, but Lagrange is not mentioned, which might be confusing.