

Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of  $\sqrt{t+1}$  gallons per minute, for  $0 \leq t \leq 120$  minutes. At time  $t = 0$ , the tank contains 30 gallons of water.

- (a) How many gallons of water leak out of the tank from time  $t = 0$  to  $t = 3$  minutes?
- (b) How many gallons of water are in the tank at time  $t = 3$  minutes?
- (c) Write an expression for  $A(t)$ , the total number of gallons of water in the tank at time  $t$ .
- (d) At what time  $t$ , for  $0 \leq t \leq 120$ , is the amount of water in the tank a maximum? Justify your answer.

(a) Method 1:  $\int_0^3 \sqrt{t+1} dt = \frac{2}{3}(t+1)^{3/2} \Big|_0^3 = \frac{14}{3}$

– or –

Method 2:  $L(t)$  = gallons leaked in first  $t$  minutes

$$\frac{dL}{dt} = \sqrt{t+1}; \quad L(t) = \frac{2}{3}(t+1)^{3/2} + C$$

$$L(0) = 0; \quad C = -\frac{2}{3}$$

$$L(t) = \frac{2}{3}(t+1)^{3/2} - \frac{2}{3}; \quad L(3) = \frac{14}{3}$$

(b)  $30 + 8 \cdot 3 - \frac{14}{3} = \frac{148}{3}$

(c) Method 1:

$$\begin{aligned} A(t) &= 30 + \int_0^t (8 - \sqrt{x+1}) dx \\ &= 30 + 8t - \int_0^t \sqrt{x+1} dx \end{aligned}$$

– or –

Method 2:

$$\frac{dA}{dt} = 8 - \sqrt{t+1}$$

$$A(t) = 8t - \frac{2}{3}(t+1)^{3/2} + C$$

$$30 = 8(0) - \frac{2}{3}(0+1)^{3/2} + C; \quad C = \frac{92}{3}$$

$$A(t) = 8t - \frac{2}{3}(t+1)^{3/2} + \frac{92}{3}$$

- (d)  $A'(t) = 8 - \sqrt{t+1} = 0$  when  $t = 63$   
 $A'(t)$  is positive for  $0 < t < 63$  and negative for  $63 < t < 120$ . Therefore there is a maximum at  $t = 63$ .

Method 1:

$$3 \begin{cases} 2 : \text{definite integral} \\ 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

– or –

Method 2:

$$3 \begin{cases} 1 : \text{antiderivative with } C \\ 1 : \text{solves for } C \text{ using } L(0) = 0 \\ 1 : \text{answer} \end{cases}$$

1 : answer

Method 1:

$$2 \begin{cases} 1 : 30 + 8t \\ 1 : -\int_0^t \sqrt{x+1} dx \end{cases}$$

– or –

Method 2:

$$2 \begin{cases} 1 : \text{antiderivative with } C \\ 1 : \text{answer} \end{cases}$$

$$3 \begin{cases} 1 : \text{sets } A'(t) = 0 \\ 1 : \text{solves for } t \\ 1 : \text{justification} \end{cases}$$

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume  $V$  of a right circular cylinder with radius  $r$  and height  $h$  is given by  $V = \pi r^2 h$ .)

- (a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?
- (b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is  $R(t) = 400\sqrt{t}$  cubic centimeters per minute, where  $t$  is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time  $t$  when the oil slick reaches its maximum volume. Justify your answer.
- (c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

- (a) When  $r = 100$  cm and  $h = 0.5$  cm,  $\frac{dV}{dt} = 2000$  cm<sup>3</sup>/min  
and  $\frac{dr}{dt} = 2.5$  cm/min.

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$

$$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.038 \text{ or } 0.039 \text{ cm/min}$$

- (b)  $\frac{dV}{dt} = 2000 - R(t)$ , so  $\frac{dV}{dt} = 0$  when  $R(t) = 2000$ .

This occurs when  $t = 25$  minutes.

Since  $\frac{dV}{dt} > 0$  for  $0 < t < 25$  and  $\frac{dV}{dt} < 0$  for  $t > 25$ ,

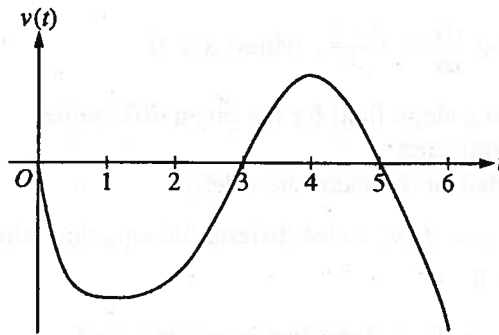
the oil slick reaches its maximum volume 25 minutes after the device begins working.

- (c) The volume of oil, in cm<sup>3</sup>, in the slick at time  $t = 25$  minutes is given by  $60,000 + \int_0^{25} (2000 - R(t)) dt$ .

$$4 : \begin{cases} 1 : \frac{dV}{dt} = 2000 \text{ and } \frac{dr}{dt} = 2.5 \\ 2 : \text{expression for } \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : R(t) = 2000 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$

$$2 : \begin{cases} 1 : \text{limits and initial condition} \\ 1 : \text{integrand} \end{cases}$$

Graph of  $v$ 

A particle moves along the  $x$ -axis so that its velocity at time  $t$ , for  $0 \leq t \leq 6$ , is given by a differentiable function  $v$  whose graph is shown above. The velocity is 0 at  $t = 0$ ,  $t = 3$ , and  $t = 5$ , and the graph has horizontal tangents at  $t = 1$  and  $t = 4$ . The areas of the regions bounded by the  $t$ -axis and the graph of  $v$  on the intervals  $[0, 3]$ ,  $[3, 5]$ , and  $[5, 6]$  are 8, 3, and 2, respectively. At time  $t = 0$ , the particle is at  $x = -2$ .

- (a) For  $0 \leq t \leq 6$ , find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.
- (b) For how many values of  $t$ , where  $0 \leq t \leq 6$ , is the particle at  $x = -8$ ? Explain your reasoning.
- (c) On the interval  $2 < t < 3$ , is the speed of the particle increasing or decreasing? Give a reason for your answer.
- (d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

- (a) Since  $v(t) < 0$  for  $0 < t < 3$  and  $5 < t < 6$ , and  $v(t) > 0$  for  $3 < t < 5$ , we consider  $t = 3$  and  $t = 6$ .

$$x(3) = -2 + \int_0^3 v(t) dt = -2 - 8 = -10$$

$$x(6) = -2 + \int_0^6 v(t) dt = -2 - 8 + 3 - 2 = -9$$

Therefore, the particle is farthest left at time  $t = 3$  when its position is  $x(3) = -10$ .

- (b) The particle moves continuously and monotonically from  $x(0) = -2$  to  $x(3) = -10$ . Similarly, the particle moves continuously and monotonically from  $x(3) = -10$  to  $x(5) = -7$  and also from  $x(5) = -7$  to  $x(6) = -9$ .

By the Intermediate Value Theorem, there are 3 values of  $t$  for which the particle is at  $x(t) = -8$ .

- (c) The speed is decreasing on the interval  $2 < t < 3$  since on this interval  $v < 0$  and  $v$  is increasing.

- (d) The acceleration is negative on the intervals  $0 < t < 1$  and  $4 < t < 6$  since velocity is decreasing on these intervals.

$$3 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{considers } \int_0^6 v(t) dt \\ 1 : \text{conclusion} \end{cases}$$

$$3 : \begin{cases} 1 : \text{positions at } t = 3, t = 5, \\ \quad \text{and } t = 6 \\ 1 : \text{description of motion} \\ 1 : \text{conclusion} \end{cases}$$

1 : answer with reason

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$

There is no snow on Janet’s driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by  $f(t) = 7te^{\cos t}$  cubic feet per hour, where  $t$  is measured in hours since midnight. Janet starts removing snow at 6 A.M. ( $t = 6$ ). The rate  $g(t)$ , in cubic feet per hour, at which Janet removes snow from the driveway at time  $t$  hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \leq t < 6 \\ 125 & \text{for } 6 \leq t < 7 \\ 108 & \text{for } 7 \leq t \leq 9. \end{cases}$$

- (a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?  
 (b) Find the rate of change of the volume of snow on the driveway at 8 A.M.  
 (c) Let  $h(t)$  represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time  $t$  hours after midnight. Express  $h$  as a piecewise-defined function with domain  $0 \leq t \leq 9$ .  
 (d) How many cubic feet of snow are on the driveway at 9 A.M.?

(a)  $\int_0^6 f(t) dt = 142.274$  or  $142.275$  cubic feet

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Rate of change is  $f(8) - g(8) = -59.582$  or  $-59.583$  cubic feet per hour.

1 : answer

(c)  $h(0) = 0$

For  $0 < t \leq 6$ ,  $h(t) = h(0) + \int_0^t g(s) ds = 0 + \int_0^t 0 ds = 0$ .

For  $6 < t \leq 7$ ,  $h(t) = h(6) + \int_6^t g(s) ds = 0 + \int_6^t 125 ds = 125(t - 6)$ .

For  $7 < t \leq 9$ ,  $h(t) = h(7) + \int_7^t g(s) ds = 125 + \int_7^t 108 ds = 125 + 108(t - 7)$ .

Thus,  $h(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq 6 \\ 125(t - 6) & \text{for } 6 < t \leq 7 \\ 125 + 108(t - 7) & \text{for } 7 < t \leq 9 \end{cases}$

3 :  $\begin{cases} 1 : h(t) \text{ for } 0 \leq t \leq 6 \\ 1 : h(t) \text{ for } 6 < t \leq 7 \\ 1 : h(t) \text{ for } 7 < t \leq 9 \end{cases}$

(d) Amount of snow is  $\int_0^9 f(t) dt - h(9) = 26.334$  or  $26.335$  cubic feet.

3 :  $\begin{cases} 1 : \text{integral} \\ 1 : h(9) \\ 1 : \text{answer} \end{cases}$

$t$ (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon ( $t = 0$ ) and 8 P.M. ( $t = 8$ ). The number of entries in the box  $t$  hours after noon is modeled by a differentiable function  $E$  for  $0 \leq t \leq 8$ . Values of  $E(t)$ , in hundreds of entries, at various times  $t$  are shown in the table above.

(a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time  $t = 6$ . Show the computations that lead to your answer.

(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of  $\frac{1}{8} \int_0^8 E(t) dt$ .

Using correct units, explain the meaning of  $\frac{1}{8} \int_0^8 E(t) dt$  in terms of the number of entries.

(c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function  $P$ , where  $P(t) = t^3 - 30t^2 + 298t - 976$  hundreds of entries per hour for  $8 \leq t \leq 12$ . According to the model, how many entries had not yet been processed by midnight ( $t = 12$ )?

(d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.

(a)  $E'(6) \approx \frac{E(7) - E(5)}{7 - 5} = 4$  hundred entries per hour

1 : answer

(b)  $\frac{1}{8} \int_0^8 E(t) dt \approx$   
 $\frac{1}{8} \left( 2 \cdot \frac{E(0) + E(2)}{2} + 3 \cdot \frac{E(2) + E(5)}{2} + 2 \cdot \frac{E(5) + E(7)}{2} + 1 \cdot \frac{E(7) + E(8)}{2} \right)$   
 $= 10.687$  or  $10.688$

3 :  $\begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \\ 1 : \text{meaning} \end{cases}$

$\frac{1}{8} \int_0^8 E(t) dt$  is the average number of hundreds of entries in the box between noon and 8 P.M.

(c)  $23 - \int_8^{12} P(t) dt = 23 - 16 = 7$  hundred entries

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

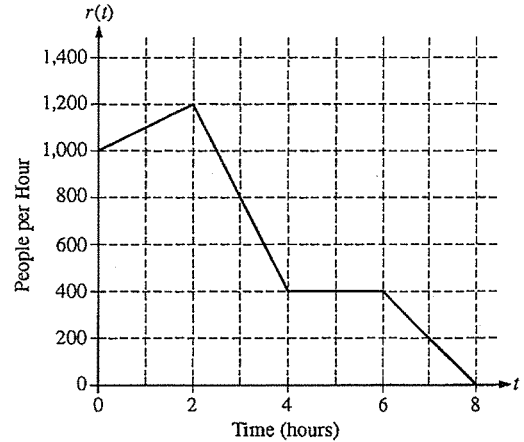
(d)  $P'(t) = 0$  when  $t = 9.183503$  and  $t = 10.816497$ .

$t$	$P(t)$
8	0
9.183503	5.088662
10.816497	2.911338
12	8

3 :  $\begin{cases} 1 : \text{considers } P'(t) = 0 \\ 1 : \text{identifies candidates} \\ 1 : \text{answer with justification} \end{cases}$

Entries are being processed most quickly at time  $t = 12$ .

There are 700 people in line for a popular amusement-park ride when the ride begins operation in the morning. Once it begins operation, the ride accepts passengers until the park closes 8 hours later. While there is a line, people move onto the ride at a rate of 800 people per hour. The graph above shows the rate,  $r(t)$ , at which people arrive at the ride throughout the day. Time  $t$  is measured in hours from the time the ride begins operation.



- (a) How many people arrive at the ride between  $t = 0$  and  $t = 3$ ? Show the computations that lead to your answer.
- (b) Is the number of people waiting in line to get on the ride increasing or decreasing between  $t = 2$  and  $t = 3$ ? Justify your answer.
- (c) At what time  $t$  is the line for the ride the longest? How many people are in line at that time? Justify your answers.
- (d) Write, but do not solve, an equation involving an integral expression of  $r$  whose solution gives the earliest time  $t$  at which there is no longer a line for the ride.

(a)  $\int_0^3 r(t) dt = 2 \cdot \frac{1000 + 1200}{2} + \frac{1200 + 800}{2} = 3200$  people

2 : { 1 : integral  
1 : answer

(b) The number of people waiting in line is increasing because people move onto the ride at a rate of 800 people per hour and for  $2 < t < 3$ ,  $r(t) > 800$ .

1 : answer with reason

(c)  $r(t) = 800$  only at  $t = 3$   
 For  $0 \leq t < 3$ ,  $r(t) > 800$ . For  $3 < t \leq 8$ ,  $r(t) < 800$ .  
 Therefore, the line is longest at time  $t = 3$ .  
 There are  $700 + 3200 - 800 \cdot 3 = 1500$  people waiting in line at time  $t = 3$ .

3 : { 1 : identifies  $t = 3$   
1 : number of people in line  
1 : justification

(d)  $0 = 700 + \int_0^t r(s) ds - 800t$

3 : { 1 :  $800t$   
1 : integral  
1 : answer