AP Central Question of the Month May 2003

Question of the Month

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LEARNING FROM MISTAKES

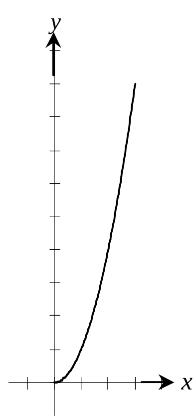
This is the first *Question of the Month* that will appear on the Calculus section of AP Central. These are not AP Exams questions, but rather questions on topics related to beginning calculus that, I hope, you will find interesting and instructive, and that you can share with your students to give them just a little deeper understanding of mathematics.

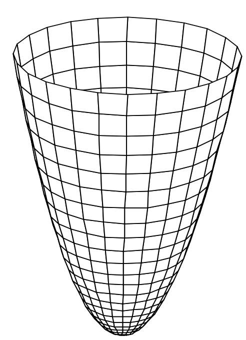
Since finding volumes by the method of Cylindrical Shells is important, but no longer on the AP Course Description, teachers often present it during the "aftermath" – the days after the AP Exams are given. So our first *Question of the Month* is about Cylindrical Shells. It is based on question 5 on the 1996 AB exam – the function was changed, but the idea remains the same. The exam question was a volume problem (which was intended to be done by the "washer" method). This was followed by an Accumulation Function / Related Rate question that used the result of the previous part.

Recently a teacher gave the original questions and one of her students found the volume using Cylindrical Shells, no problem there. But then, using Shells, the related rate computation came out very wrong. The key to understanding the mistake is understanding what you are actually finding when you work with Cylindrical Shells.

Here's our situation and your problem for this month:

> A water tank has the shape shown below, obtained by rotating the curve $y = x^2$ from $x = 0$ to $x = 3$ around the y-axis, where x and y are measured in feet. Water flows into the tank at the rate of 5 cubic feet per minute.





(A) Find the volume of water in the tank using the "Washer" method. Find the volume of water in the tank using the method of Cylindrical Shells. Indicate units of measure for both. Compare your answers. (They should be the same of course.)

(B) Let h be the depth of water in the tank. Using a definite integral, write two functions that give the volume of water in the tank as a function of h, one, $V_w(h)$, using the "Washer" method and the other, $V_s(h)$, using the method of "Cylindrical Shells."

(C) How fast is the depth of water in the tank increasing when $h = 2$? Indicate units of measure. Again do this part using both the "Washer" and "Cylindrical Shells" method. Compare your answers. If they are the same, congratulations! You may take the rest of the day off. If they're not the same, find out why not. (Hint: Look at the picture you used for the "Shell" method in (b); how is the water accumulating?)

Do all three parts of the question before checking the answer and explanation!

answer and explanation = Link Answer and Explanation.

End Question

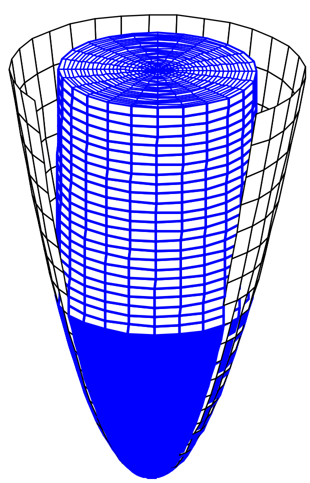
Answers and Explanation:

(a) Washers: $V_w = \pi\int_0^9 x^2\,dy = \pi\int_0^9 y\,dy = 40.5\pi$ cubic feet.

Shells: $V_s = 2\pi\int_0^3 x(9-y)\,dx = 2\pi\int_0^3 x(9-x^2)\,dx = 40.5\pi$ cubic feet.

(b) Washers: $V_w(h) = \pi\int_0^h x^2\,dy = \pi\int_0^h y\,dy$

Here's the interesting part: For the volume by shells, $V_s\left(h\right) \neq 2\pi\int_0^{\sqrt{h}} x\left(9-x^2\right)dx$. The temptation is to do the "Shell" part the same way as the "Washer" part. This doesn't work because of the way shells are set up. This integral gives the volume of water currently in the tank plus the water in a cylinder "sitting on top" of the water in the tank. (The cylinder has a radius of $\sqrt{h}$ and a height of $9-h$.) This is as if the water were filling an expanding cylinder running from the top of the tank down to the current water level. Water does not behave that way when you pour it into a tank. When the tank is completely full the volume is the same, but it is not the same while the tank is filling.



[Caption:] The partially filled tank?

There are two correct functions using the Cylindrical Shells approach for the accumulating volume. First, the variable height shows up in the integrand as well as the limit of integration:

$$V_{s1}(h) = 2\pi\int_0^{\sqrt{h}} x(h-y)\,dx = 2\pi\int_0^{\sqrt{h}} x\left(h-x^2\right)dx\,.$$

Or the "extra" cylinder above the current level of water can be subtracted:

$$V_{s2}(h) = \left[2\pi\int_0^{\sqrt{h}} x\left(9-x^2\right)dx\right] - \left[\pi\left(\sqrt{h}\right)^2(9-h)\right].$$

(c) By Washers using the Fundamental Theorem of Calculus: Recall that $V_w(h) = \pi\int_0^h y\,dy$, so therefore $\frac{dV_w}{dt} = \pi h\frac{dh}{dt}$, and when h = 2 and $\frac{dV_w}{dt} = 5, \frac{dh}{dt} = \frac{5}{2\pi}$ feet per minute.

By Shells: First, if you reasoned

$$V_s(h) = 2\pi\int_0^{\sqrt{h}} x\left(9-x^2\right)dx$$

$$\frac{dV_s}{dt} = 2\pi\sqrt{h}\left(9-\left(\sqrt{h}\right)^2\right)\left(\frac{1}{2\sqrt{h}}\right)\frac{dh}{dt}$$

reread "Here's the interesting part" above again. Otherwise, using the Fundamental Theorem of Calculus:

$$\begin{aligned}\frac{dV_{s2}}{dt} &= \frac{d}{dt}\left[2\pi\int_0^{\sqrt{h}} x\left(9-x^2\right)dx - \pi\left(9h-h^2\right)\right]\\ &= \left[2\pi\left(\sqrt{h}\right)\left(9-\left(\sqrt{h}\right)^2\right)\left(\frac{1}{2}h^{-1/2}\right) - \pi(9-2h)\right]\frac{dh}{dt}\\ &= [9\pi - \pi h - 9\pi + 2\pi h]\frac{dh}{dt}\\ &= \pi h\frac{dh}{dt}\end{aligned}$$

Then, when h = 2 and $\frac{dV_{s2}}{dt} = 5, \frac{dh}{dt} = \frac{5}{2\pi}$ feet per minute.

The function $V_{s1} = 2\pi\int_0^{\sqrt{h}} x\left(h-x^2\right)dx$ *cannot* be differentiated directly using the Fundamental Theorem of Calculus because the independent variable h appears in the integrand. More advanced methods are required.

Finally, the easy way to do the problem is to evaluate any one of the integrals in (b): $V_w(h) = V_{s1}(h) = V_{s2}(h) = \frac{\pi}{2}h^2$. Then $\frac{dV}{dt} = \pi h\frac{dh}{dt}$ and the rest is easy.

Doug Kuhlmann, of Phillips Academy Andover, was the first to give a cogent explanation of the original “mistake” when the question was first asked on the AP Calculus Electronic Discussion Group. Thank you Doug!