Improper Integrals and Proper Areas

By Lin McMullin

Recently a teacher wrote to the AP Calculus EDG with a question concerning the improper integral

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx \\ &= \lim_{b\to\infty} \arctan(x)\Big|_0^b \\ &= \lim_{b\to\infty}\left(\arctan(b) - \arctan 0\right) \\ &= \tfrac{\pi}{2} - 0 \\ &= \tfrac{\pi}{2}\end{aligned}$$

His (quite perceptive) student pointed out that the range of the $\arctan(x)$ is arbitrarily restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The student asked if using some other range would affect the answer to this problem.

The short answers is no, the answer is the same. The reason is that in that case the $\arctan 0$ term is not zero, but the difference is still $\frac{\pi}{2}$. For example, if the section of the graph of $y = \tan(x)$ in the interval $\left(\frac{5\pi}{2}, \frac{7\pi}{2}\right)$ were reflected over the line $y = x$, then that branch of the inverse tangent relation may be used to define the range of the inverse: $\frac{5\pi}{2} < \arctan(x) < \frac{7\pi}{2}$. Then, in this case

$$\lim_{b\to\infty}\left(\arctan(b) - \arctan 0\right) = \tfrac{7\pi}{2} - 3\pi = \tfrac{\pi}{2}$$

The same value as above.

Now that's all pretty straightforward, but there are other things going on here. Here's another way to approach the student's question. Besides seeing that only one answer is possible, this may be a way to help your students understand why improper integrals are defined in terms of limits.

The original definite integral represents the area in the first quadrant between the graph of $y = \frac{1}{1+x^2}$ and the *x*-axis. Let's consider the accumulation function that gives the area of that region between the *y*-axis and the vertical line at finite values of *x*. The area function is

$$A(x) = \int_0^x \frac{1}{1+t^2}\,dt$$

As *x* moves from the origin to the right along the *x*-axis the area increases. For example, $A(1.5) \approx 0.98279$ is shaded in Figure 1.

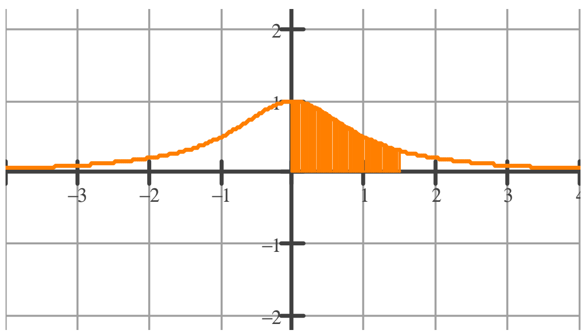

Figure 1

Pretending for a moment that we don't know the antiderivative, the area function can be graphed on a calculator by entering Y1(X) = fnInt(1/(1 + T^2), T, 0, X) . Figure 2 shows the graph of the area function.

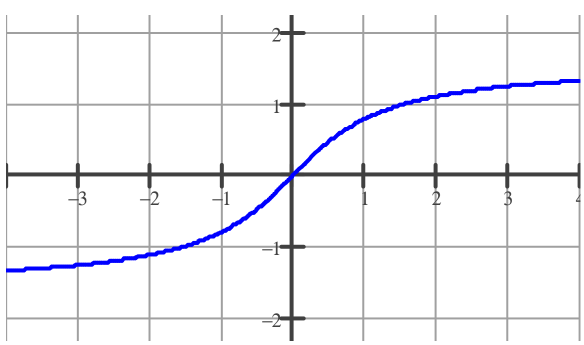

Figure 2

If you're good at recognizing graphs, you will suspect this is $y = \tan^{-1}(x)$, the arctangent function. But what's really interesting here is that whatever function it is, it seems to have a horizontal asymptote: the area is approaching a finite limit as $x \to \infty$.

The connection with improper integrals is obvious. By definition

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{x\to\infty} A(x)$$

The improper integral is defined as the limit of the area function! The area function has a horizontal asymptote as $x \to \infty$ and therefore approaches a finite value. The unbounded region has a finite area.

Other things you may want to discuss on the way by:

1. What is the relationship between a function in red, and its antiderivative in blue shown in Figure 3. What does the red graph tell you about the blue graph? What does the blue graph tell you about the red?

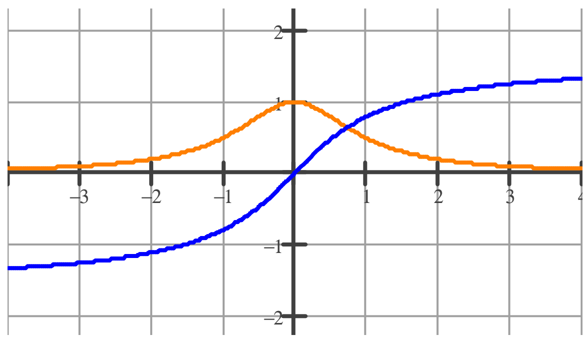

Figure 3

2. Horizontal asymptotes and derivatives: As $x \to \infty$ the graph of the function (red) gets flatter, so its antiderivative's slope approaches zero. Where a derivative (red) approaches zero as $x \to \infty$, the function (blue) has a horizontal asymptote, and conversely. (Likewise as $x \to -\infty$.)

3. Discuss the part of the definite integral (blue) to the left of the *y*-axis in Figure 3. Why is the "area" of the region between the red graph and the *x*-axis negative?