The First Week of AP Calculus

by
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Graphs were done using PSMathGraphsII, written for older Macintosh operating systems by John Jacob of Marin College, San Rafael. CA, email John.Jacob@marin.edu

Calculus involves just four concepts:

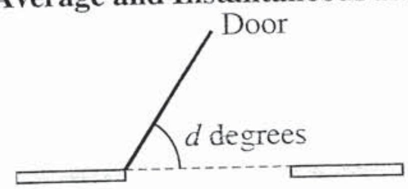
- Limits
- Derivatives
- Integrals
- Integrals

Graphing calculators and "anatomically correct" computer graphing allow students to learn about the first three concepts starting on Day 1, not after weeks of precalculus review.

Here's how I do this with my students in either AB or BC Calculus.

>>>——> The power of saying,
"You recall..."

Day 1: Average and Instantaneous Rates



A door with a closer swings open then closes. At time t seconds, it forms and angle of d degrees with the wall.

a. Sketch a reasonable graph.

b. If $d(t) = 200t \cdot 2^{-t}$, plot on your grapher.

c. Find the average rate of change of d :
- From 1 to 1.1 seconds ____________
- From 1 to 1.01 seconds ____________
- From 1 to 1.001 seconds ____________
- From 1 to 1.0001 seconds ____________

d. The **instantaneous rate of change** of $d(t)$ with respect to t is the **limit** of the average rates as the length of the time interval approaches 0.

- Conjecture: Instantaneous rate = _______

The instantaneous rate of change is called the **derivative** of $d(t)$ with respect to t.

Solutions to Door Closer Problem, Day 1

(For the actual exploration the students do in class, see Exploration 1-1a from KCP/KH Calc.)

a. Any graph starting at the origin, rising to a max, then falling toward zero is reasonable. (Review: Degrees on the *vertical* axis because angle *depends* on time. Label the axes.)

b. (Review: Grapher use, "$f(x)$" terminology.)

c. • $d(1) = 100$ and $d(1.1) = 102.63362...$
So the door opened 2.63362... degrees in 0.1 s.

Average rate $= \dfrac{2.6336...}{0.1} = \underline{26.3362...\text{ deg/s}}$

(Avoid use of any preconceived formula!)

- From 1 to 1.01 seconds: <u>30.2342... °/s</u>
- From 1 to 1.001 seconds: <u>30.6400... °/s</u>
- From 1 to 1.0001 seconds: <u>30.6807... °/s</u>

(Review: "Ellipsis" form for unrounded values, and round only the *final* answer.)

d. Any conjecture *greater* than 30.6807... .
Exact derivative is <u>30.6853...</u> (non-integer).
(A *product* with an *exponential function!*)
(Review: Meaning of "conjecture.")
• Conclusion: Derivative — *Inst.* rate of change

Homework, first night

Board Price Problem: The price of a 2 by 6 board depends on its length. Assume that the price is given by the cubic function

$$y = 0.2x^3 - 4.8x^2 + 80x$$

where y is in cents and x is in feet.

a. Find prices of 5 ft, 10 ft, and 20 ft boards.

b. Find the average rate of change of price in cents per foot as x goes from
- 5 to 5.1 feet
- 5 to 5.01 feet
- 5 to 5.001 feet

c. The average number of cents per foot in part b is approaching an integer as the change in x approaches zero. This integer is called the "—?— rate of change." What goes in the blank?

d. The number in part c is an example of two of the four concepts of calculus. Which two?

e. Estimate the instantaneous rate of change in price for 2 by 6 boards 10 ft. and 20 ft. long.

f. Why is the instantaneous rate lower for a 10-foot board than for either a 5-foot or 20-foot?

Answers to Board Price Problem

a. $x = 5$: $y = 305$ (mathematical world ans.)
5-ft board costs <u>$3.05</u> (real-world answer)
10-ft board costs <u>$5.20</u>
20-ft board costs <u>$12.80</u>

b. Av. rate, 5 to 5.1: <u>46.822 ¢/ft</u>
Av. rate, 5 to 5.01: <u>46.9820... ¢/ft</u>
Av. rate, 5 to 5.001: <u>46.9982... ¢/ft</u>

c. The average rates approach <u>47 ¢/ft</u>.
The word is <u>instantaneous</u> rate of change.

d. Concepts: **limit** and **derivative**

e. Instantaneous rate at 10 ft: <u>44 ¢/ft</u>
Instantaneous rate at 20 ft: <u>128 ¢/ft</u>

f. "Buying in quantity" usually results in lower prices. But very long boards are scarce, so the price per foot actually increases.

(Review: Math. world vs. real world answers. Interpretation of answers.)

• Conclusion reinforced: A **derivative** is an *instantaneous* rate of change, and is the **limit** of average rates of change.

Day 2: Recall graphs of "familiar" functions

(See Exploration 1-2a from KCP/KH Calculus.)

Sketch the graph of:

a. $f(x) = 3^{-x}$

b. $f(x) = \sin \frac{\pi}{2}x$

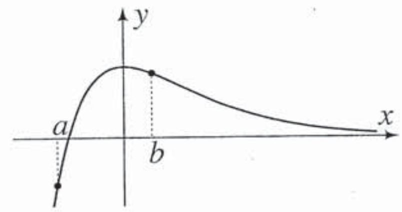
c. $f(x) = x^2 + 2x - 2$

d. $f(x) = \frac{1}{x}$

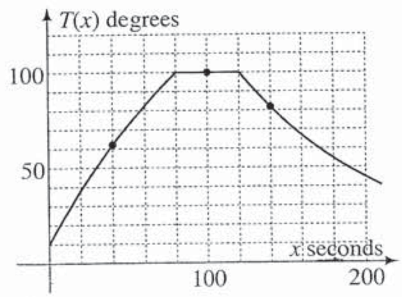
(Review: Graphs of various kinds of function. Horizontal dilation factor in part b.)

Homework for Day 2 includes problems like:

1. Is the function graphed increasing or decreasing at the points shown? Is the rate of change relatively fast or relatively slow?



2. *Boiling Water Problem:* Water in a kettle is at $T(x)$ degrees when x seconds have elapsed since you started heating it (see graph).

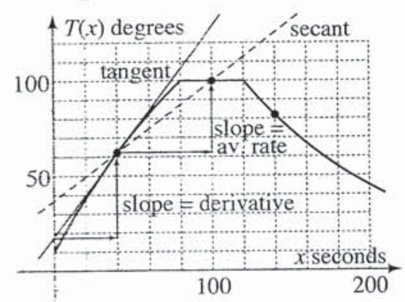


Estimate the instantaneous rate of change of temperature at $x = 40$, 100, and 140 s. Units?

Solutions, Day 2 Sample Homework Problems

1. a. <u>Increasing relatively fast.</u>
b. <u>Decreasing relatively slowly.</u>

2. • Average rate between 40 s and 100 s
= slope of *secant* line.



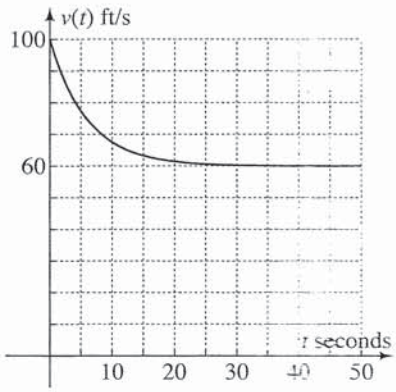
• Instantaneous rate at 40 s
= *limit* of slope of secant line
as the other x-value approaches 40
= slope of *tangent* line at $x = 40$.

Run = 40 − 0 = 40 s
Rise = $T(40) - T(0) \approx 62 - 18 = 44$ degrees
Inst. rate $\approx \frac{44}{40} =$ <u>1.1 deg/s</u> (exact: 1.1075...)

At 100 s, <u>0 deg/s</u>, at 140 s, <u>−0.8220... deg/s</u>
(Review: pos. & neg. slope, piecewise function)

Day 3: Introduction to Definite Integrals

(See Exploration 1-3a from KCP/KH Calculus.)



After passing a truck, your car's speed $v(t)$ decreases from 100 ft/s toward 60 ft/s (graph)

1. How far do you travel in the time interval [30, 50]? Shade the rectangular region whose area equals this number.

2. Shade the region whose area equals the distance you travel in the interval [0, 20]. Estimate this distance by counting squares on the graph. (What distance does each small square represent?)

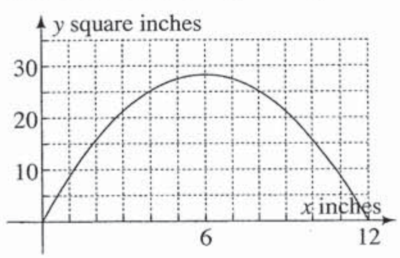
Day 3, continued

The answer to Problem 2 is a product of velocity and time, and thus has units

$$\text{Velocity} \times \text{Time} = \frac{\text{ft}}{\text{s}} \times \text{s} = \text{ft}$$

This product is called a **definite integral** of velocity with respect to time from 0 to 20 s.

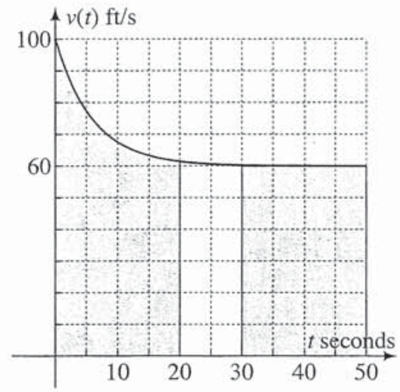
3. The graph below shows the cross-sectional area of a football as a function of the distance x from one of its ends. Find an estimate for the definite integral of cross-sectional area with respect to distance from the end. What physical quantity does this integral represent?



• Conclusion: Physically, a definite integral *is* a product of x and y, where y might *vary*.

(Review: Interval notation, dimensional analysis)

Solutions, Day 3



1. Rate × Time = 60 × 20 = <u>1200 feet</u>

2. Approximately 28.6 small squares, each representing (10)(5) = 50 feet.
Dist. ≈ (28.6)(50) = <u>1430 ft</u> (1431.3207...)

3. Approximately 45.2 small squares, each representing (5)(1) = 5 ft^3
Integral ≈ (45.2)(5) = 226 ft^3 (226.1946...)
The <u>volume</u> of the football!

• Conclusion: Graphically, a definite integral *equals* the area of a region under a graph, and can *represent* a length or a volume.

Homework for Day 3

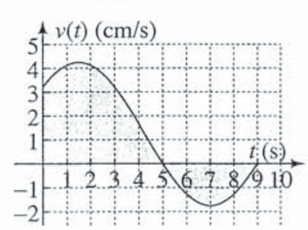
1. Integral problems by counting squares on given graphs, or by first plotting graphs of given functions on graph paper.

2. Finish with :

Negative Velocity Problem: Velocity differs from speed because it can be negative. If the velocity is negative, the moving object's *displacement* from a certain fixed position is decreasing. The figure shows the velocity of a moving object. Estimate:

a. The object's net displacement from the starting point after 9 seconds.

b. The total distance the object travels back and forth in the 9-second time period.



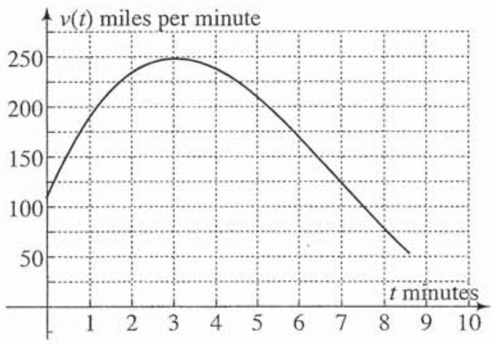
• Displ. = integral fr. 0 to 5 + integral fr. 5 to 9
≈ 15.0 + (−4.6) = <u>10.4 cm from starting point</u>

• Distance ≈ |15.0| + |−4.6| = <u>19.6 cm traveled</u>

Day 4: Numerical Approx. of Def. Integrals

(See Exploration 1-4a from KCP/KH Calculus.)

Rocket Problem: At time $t = 0$ minutes, Ella Vader fires her rocket engines. The figure shows her velocity, $v(t)$ miles per minute versus t.



1. Approximately how far does she go from time $t = 0$ to $t = 8$ minutes?

2. Ella figures that her velocity is given by

$$v(t) = t^3 - 21t^2 + 100t + 110$$

By finding values of $v(t)$ for several values of t, show that the graph agrees with this equation.

Solutions for Ella's Rocket Problem

1. Approx. 60.8 "squares,"
each representing (1)(25) = 25 ft
Total distance ≈ (60.8)(25) = 1520 ft

2.

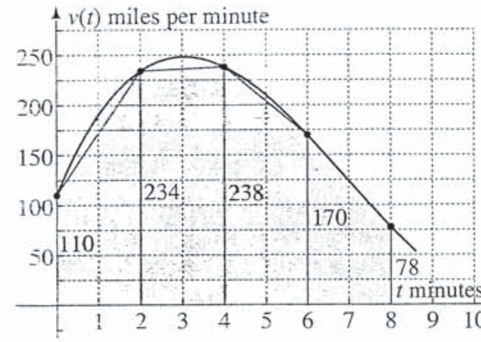
t	$v(t)$
0	110
2	124
4	238
6	170
8	78

The table values agree with the graph. (Review: Table feature of grapher)

Rocket Problem, Continued:

3. Divide the region under the graph into four strips of equal width. Draw four trapezoids in these strips who parallel sides are values of $v(t)$, and thus whose areas are approximately the areas of the strips. Find an estimate of the distance in Problem 2 by summing the areas of the trapezoids.

Rocket Problem Concluded



• Trap. area = 344 + 472 + 408 + 248 = 1472
Underestimates b/c trapezoids are *inscribed.*

4. Download TRAPRULE program, show that it gives 1472 for four trapezoids.

5. Estimate the **limit** the areas approach as the number of trapezoids increases. This limit is the *exact* value of the definite integral.

n	Area
4	1472 (agrees with Problem 3)
20	1518.08
100	1519.9232
1000	1519.999232 Conjecture: Limit = 1520.

Conclusions:

• The graphing calculator and computer graphics allow calculus to start on Day 1.

• The numerical and graphical approach gives students an intuitive feeling for the meanings of limit, derivative, and definite integral that gets lost if the students simply memorize formulas.

• Starting calculus with precalculus review:
   a. Takes time that could be used later on.
   b. Review, before needed, is forgotten again.
   c. Lulls students into a false sense of security.
   d. Discourages the weaker students. (Ask!)

• The examples used here are beyond the reach of simple algebraic calculus formulas the students may have encountered in precalculus.

• A derivative is an instantaneous rate of change or the limit of average rates of change.

• A definite integral is the product of x and y even if y happens to vary. The integral equals the area of the region under the graph, and is the limit of the sum of the areas of trapezoids.

Exam: Write the one most important thing you learned as a result of attending this session.