22

"TYPE" PROBLEMS

The AP calculus exams contain fresh carefully thought out often cleaver questions. This is especially true for the free-response questions. The topics and style of the questions are similar from year to year. By style I mean whether the stem presents the given information as an equation, a graph or a table of values. Most of the topics can be and have been presented in each of the styles. With that in mind this chapter will discuss the common types of questions on the exams. Some will be discussed based on their style and other by the calculus topic being tested.

Some of the questions test topics that are usually found in one or several contiguous sections of most textbook. These include the area-volume questions, differential equations, and in BC the parametric equation-vector question, and the power series question.

The others tend to draw questions from diverse parts of the book taught at different times of the year, yet based on the same stem. Students need to be aware of this and be ready to shift gears in the middle of the question. They may not be used to this since textbook questions tend to be about the topic in that section and not drawn from previous sections.

Using released exam questions during the year and especially as part of the review at the end of the year will help students draw their knowledge together and do their best on the exam. Another way of developing this skill in students is to make all tests and quizzes cumulative from the beginning of the year. Always include a few from previous units on each test.

Keep in mind that each part of a free-response question can be re-written as a multiple-choice question. You can also use them as shorter open end questions on your tests.

What follows is a discussion of the 10 types of questions.

Type 1: Rate & Accumulation

These questions are often in a "real" context with a lot of words describing a situation in which some things are changing. There are usually two rates given acting in opposite ways. These are also known as "in-out" questions.Students are asked about the change that the rates produce over some time interval either separately or together. The rates may be given in an equation, a graph or a table.

The question almost always appears on the calculator allowed part of the free-response exam and the rates are often fairly complicated functions. If they are on the calculator allowed section, students should store the equations in the equation editor of their calculator and use their calculator to do any equation solving, integration, or differentiation that may be necessary.

Shorter questions on this concept appear in the multiple-choice sections. As always, look over as many questions of this kind from past exams.

What should students be able to do?

- Be ready to read and apply; often these problems contain a lot of writing which needs to be carefully read and interepreted.
- Recognize that the word "rate" means that a derivative is given even though the word "derivative" may not appear.
- Recognize a rate from the units given without the words "rate" or "derivative."
- Find the change in an amount by integrating the rate. The integral of a rate of change gives the amount of change (FTC): $\int_a^b f'(t)\,dt = f(b) - f(a)$.
- Find the final amount by adding the initial amount to the amount found by integrating the rate. The final accumulated amount is $f(t) = f(t_0) + \int_{t_0}^{t} f'(x)dx$, where $t = t_0$ is the initial time, and $f(t_0)$ is the initial amount.
- Understand the question. It is often not necessary to as much computation as it seems at first.
- Use FTC to differentiate a function defined by an integral. Remember, the integrand is the derivative of the integral.
- Explain the meaning of a derivative or its value in terms of the context of the problem.
- Explain the meaning of a definite integral or its value in terms of the context of the problem.
- Justify your answer.

AP Questions

(FR) **2007** AB *2*
(FR) **2009** AB *3*
(FR) **2010** AB *1*
BC *1*
(MC) **2012** AB *81*
BC *81*
(FR) **2013** AB *1*
BC *1*

- Store functions in their calculator recall them to do computations on their calculator.
- If the rates are given in a table, be ready to approximate an integral using a Riemann sum or by trapezoids.
- Do a max/min or increasing/decreasing analysis.

These topics are discussed in chapters 7, 14 and 20 of this book.

Type 2: Particle Moving on a Line

These questions may give the position, the velocity, or the acceleration along with an initial condition. Students may be asked about the motion of the particle: its direction, when it changes direction, when it is farthest left or right, when it turns around, how far it travels its position at a certain time, etc. Speed, the absolute value of velocity, is also a common topic.

The particle may be a "particle," a person, a car, etc. The position, velocity, or acceleration may be given as an equation, a graph or a table. There are a lot of different things students may be asked to find. While particles don't really move in this way, the question is a versatile way to test a variety of calculus concepts.

AP Questions

(FR) **2003** AB *5*
(FR) **2006** AB *4*
(FR) **2008** AB *4*, BC *4*
(FR) **2009** AB *1*, BC *1*
(MC) **2012** AB *16*, *28*, *79*, *83*, *89*
BC *2*, *89*
(FR) **2013** AB *2*

What should students be able to do?

- Solve an initial value differential equation problems: given the velocity (or acceleration) with initial condition(s) find the position (or velocity).
- Distinguish between position at some time, and the total distance traveled during the time (displacement).
  - The total distance traveled is the definite integral of the speed (absolute value of velocity): $\int_a^b |v(t)|dt$
  - The net distance (displacement) is the definite integral of velocity: $\int_a^b v(t)dt$
  - The final position is the initial position plus the net change in distance from $x = a$ to $x = t$: $s(t) = s(a) + \int_a^t v(x)dx$ Notice that this is an accumulation function equation.
- Find the speed at a particular time.
- Find average speed, velocity, or acceleration. (Average rate of change, average value of a function)
- Determine the speed and whether it is increasing or decreasing.
  - If at some time, the velocity and acceleration have the *same* sign then the speed is increasing.
  - If they have *different* signs the speed is decreasing.
  - If the velocity graph is moving away from (towards) the t-axis the speed is increasing (decreasing).
- Use a difference quotient to approximate derivative from a graph or table. (Show a quotient even if the denominator is 1.)
- Approximate velocity or acceleration from a graph of table.

- Approximate an integral using a Riemann sum or trapezoid sum with values from a table.
- Determine units of measure.
- Interpret meaning of a derivative or a definite integral in context of the problem.
- Justify your answer.

These topics are discussed in chapter 9 with ideas from chapters 11, 14 and 20 of this book.

Type 3: Interpreting Graphs

The long name is "Here's the graph of the derivative, tell me things about the function."

Most often students are given the graph identified as the derivative of a function. There is no equation given and it is not expected that students will write the equation; rather, students are expected to determine important features of the function or its graph directly from the graph of the derivative. They may be asked for the location of extreme values, intervals where the function is increasing or decreasing, concavity, etc. They may be asked for function values at points, found by using the areas of the regions on the derivative's graph.

The graph may be given in context and student will be asked about that context. The graph may be identified as the velocity of a moving object and questions will be asked about the motion and position.

Less often the function's graph may be given and students will be asked about its derivatives.

What should students be able to do?

- Read information about the function from the graph of the derivative. This may be approached using derivative techniques or techniques using the definite integral.
- Find and justify extreme values (1st and 2nd derivative tests, closed interval test, etc.)
- Find where the function is increasing or decreasing.
- Find points of inflection.
- Write an equation of tangent line.
- Evaluate Riemann or trapezoidal sums from geometry of the graph or from a table
- Evaluate integrals from areas of regions on the graph.
- Understand that the function, $g(x)$, maybe defined by an integral where the given graph is the graph of the integrand, $f(t)$. So students should know that if $g(x) = g(a) + \int_a^x f(t)\,dt$, then $g'(x) = f(x)$ and $g''(x) = f'(x)$
- Justify your answer.

AP Questions

(FR) **2005** AB *5*
(FR) **2010** AB *5*
(FR) **2011** AB *4*, BC *4*
(MC) **2012** AB *15*, *17*, *76*, *80*, *85*, *87*
BC *15*, *18*, *76*, *78*, *80*, *88*
(FR) **2012** AB *3*, BC *3*
(FR) **2013** AB *4*, BC *4*

The ideas and concepts that can be tested with this type question are numerous. The type appears on the multiple-choice exams as well as the free-response. They have accounted for about 20% of the points available on recent tests. It is very important that students are familiar with all of the ins and outs of this situation.

These topics are discussed mainly in chapters 9 12, 14 and 20 of this book.

Type 4: Area – Volume

Given equations that define a region in the plane students are asked to find its area, the volume of the solid formed when the region is revolved around a line, or used as a base of a solid with regular cross-sections. These standard applications of the integral have appeared on every exam since the first AB exam in 1969 and all but one year on the BC exam.

If this questions appears on the calculator active section:

- It is expected that the definite integrals will be evaluated on a calculator. Students should write the definite integral with limits on their paper and put its value after it. It is *not* required to give the antiderivative and if students give an incorrect antiderivative they may lose credit even if the final answer is (somehow) correct.
- Students should enter the equations in the equations editor and then recall them for computations.
- Students may have to find the point(s) of intersection of the functions and use them as the limits of integration.
  - Students should write the coordinates of the intersection(s) on their paper and assign a letter to each. They may them use the letter as a limit of integration from then on.
  - Students should store and recall them on the calculator for computations. This avoids copy errors and round off errors.
  - Don’t round. If rounded or truncated limits of integration result in a final value that is not correct to three places after the decimal point, the final answer is wrong and will lose a point.

What should students be able to do?

- Find the point(s) of intersection of two graphs.
- Find the area of the region between the graph and the axis or between two graphs.
- Find the volume when the region is revolved around a line, not necessarily an axis, by the disk/washer method. (Shell method is *never necessary*, but is eligible for full credit if properly used.)
- Find the volume of a solid with regular cross-sections whose base is the region between the curves. See especially 2008 (FR) AB1/BC1 (d) and 2009 (FR) AB 4(b)
- Find the equation of a horizontal or vertical line that divides the area or volume of the region in half. This involves setting up and solving an

AP Questions

(FR) **2008** AB *1*, BC *1*
(FR) **2009** AB *4*
(FR) **2010** AB *5*
(MC) **2012** AB 10, 92, BC *87*
(FR) **2012** AB *2*

integral equation where the limit is the variable for which the equation is solved. See 2012 (FR) AB 2(d)

- For BC only – find the area of a region bounded by polar curves.

These topics are discussed mainly in chapters 11, 12 and 15 of this book.

Type 5: Table Questions

Tables may be used to test a variety of ideas in calculus including analysis of functions, accumulation, position-velocity-acceleration, theory, and theorems among others. Numbers and working with numbers is part of the Rule of Four and table problems are how this is tested.

What should students be able to do?

- Find the average rate of change over an interval or approximate the derivative using a difference quotient. Use the two values closest to the number being approximated. This amounts to finding the slope.
- Use Riemann sums (left, right, midpoint) or a trapezoidal approximation to approximate the value of a definite integral using values in the table (typically with subintervals of uneven length). The Trapezoidal Rule, *per se*, is not tested; it is expected that students will add the areas of several trapezoids without reference to a formula.
- Use a table values to find numbers guaranteed by certain theorems such as average value, average rate of change, Rolle's Theorem, the Mean Value Theorem, the Intermediate Value Theorem, and find inverses. (See 2007 AB 3 which has a mean score of less than 1out of 9 points)
- Use a table values to evaluate derivatives by the product, quotient and chain rules.
- If the question is presented in some context answers should be in that context.

Do's and Don'ts

- Do: Remember that you do not know what happens between the values in the table unless some other information is given.
- Don't assume that the largest number in the table is the maximum value of the function, or the smallest value is the minimum.
- Do: Show what you are doing even if you can do it in your head. If you're finding a slope, show the quotient even if the denominator is 1. This is because you are required to show your work. A "bald" answer, one with no work, even if correct may not receive credit.
- Don't do arithmetic. A long expression consisting entirely of numbers such as you get when doing a Riemann sum, does not need to be simplified in any way. If you simplify a correct answer incorrectly, you will lose credit. It is okay to leave things like $\cos(2)$ or even $1 + 1$.
- Do not: Use a calculator to find a regression equation and then use that to answer parts of the question. While regression is good mathematics, regression equations are not one of the four things students may do with

AP Questions

(FR) **2007** AB *3*
(FR) **2010** AB *2*, BC *2*
(FR) **2011** AB *2*, BC *2*
(MC) **2012** AB *8*, *86*, *91*, BC *8*, *81*, *86*
(FR) **2013** AB *3*, BC *3*
(FR) **2014** AB *4*, BC *4*

their calculator. Regression gives only an approximation of our function. The exam is testing whether students can work with numbers.

Shorter questions on this concept appear in the multiple-choice sections. As always, look over as many questions of this kind from past exams.

These topics are discussed in chapters 7 – 15 of this book.

Type 6: Differential Equations

The actual solving of the differential equation is usually the main part of the problem, but it is accompanied by a question about its slope field or a tangent line approximation. BC students may also be asked to approximate using Euler's Method.

AP Questions

(FR) **2002** BC *5*
(FR) **2007** AB *4b*
(FR) **2008** AB *5*, BC *6*
(MC) **2008** AB *22*, *27*
BC *24*, *27*
(FR) **2010** AB *6*
(MC) **2012** AB *23*, *25*
BC *12*, *16*, *23*
(FR) **2012** AB *5*, BC *5*
(FR) **2013** AB *6*, BC *5*

What should students be able to do?

- Find the *general solution* of a differential equation using the method of separation of variables (this is the *only* method tested).
- Find a *particular solution* using the initial condition to evaluate the constant of integration – initial value problem (IVP).
- Understand that proposed solution of a differential equation is a function (not a number) and if it and its derivative are substituted into the given differential equation the resulting equation is true. This may be part of doing the problem even if solving the differential equation is not required (See (FR) 2007 AB 4b)
- Growth-decay problems.
- Draw a slope field by hand.
- Sketch a particular solution on a given slope field.
- Interpret a slope field.
- Use the given derivative to analyze a function such as finding extreme values
- For BC only: use Euler's Method to approximate a solution.
- For BC only: use the method of partial fractions to find the antiderivative after separating the variables.
- For BC only: understand the logistic growth model, its asymptotes, meaning, etc. The exams have never asked students to actually solve a logistic equation IVP.

Large parts of the BC questions are often suitable for AB students and contribute to the AB subscore of the BC exam. AB teachers may adapt these for their students.

Shorter questions on this concept appear in the multiple-choice sections.

These topics are discussed mainly in chapter 16 of this book.

Type 7: Occasional Topics

These two topics appear now and then on the free-response exams. Sometimes they appear as a full questions, other times as part of a question.

Implicit Differentiation

These questions may ask students to find the first or second derivative of an implicitly defined relation. The first derivative may be given and students are required to show that it is correct. (This is because without the correct derivative the rest of the question cannot be done.) The follow-up is to answer some question about the function such as finding an extreme value, second derivative test, or find where the tangent is horizontal or vertical.

Other times implicit differentiation may be only a part of a question. Implicit differentiation questions also appear on the multiple-choice sections

What should students be able to do?

- Implicit differentiation: know how to find the first derivative of an implicit relation using the product rule, quotient rule, the Chain rule, etc.
- Know how to find the second derivative, including substituting for the first derivative.
- Know how to evaluate the first and second derivative by substituting both coordinates of the point and the value of the derivative there. (Note: If all that is needed is the numerical value of the derivative then the substitution is often easier if done before solving for dy/dx or d^2y/dx^2)
- Analyze the derivative to determine where the relation has horizontal and/ or vertical tangents.
- Work with lines tangent to the relation, find their equations etc.
- Find extreme values. It may also be necessary to show that the point where the derivative is zero is actually on the graph.

AP Questions

(FR) **2004** AB *4*
(FR) **2008** AB *6* (form B)
(MC) **2012** AB *27*

These topics are discussed mainly in chapter 8 of this book.

Related Rates

Derivatives are rates and when more than one variable is involved the relationships among the rates can be found by differentiating with respect to time. The time variable may not appear in the equations. These questions appear occasionally on the free-response sections often as part of a longer question. A simple question may appear in the multiple-choice sections.

What should students be able to do?

- Set up and solve related rate problems.
- Know how to differentiate with respect to *time*
- Interpret the answer including units.

AP Questions

(FR) **2001** AB *5*
(FR) **2002** AB *6*
(FR) **2003** AB *5*, BC *5*
(FR) **2008** AB *3a*
(MC) **2012** AB *88* BC *88*

These topics are discussed mainly in chapter 10 of this book.

Type 8: Parametric and Vector Questions (a BC topic)

Rather than the AB question about a particle moving on a line (Type 2), the BC exams have things moving in the plane. The position or velocity is given as a pair of parametric equations or a vector. Students are asked questions about the motion, the velocity, acceleration, and speed.

Vectors may be written using parentheses, ( ), or pointed brackets, $\langle\ \rangle$, or even $\vec{i}$-$\vec{j}$ form. The pointed brackets seem to be the most popular right now, but any notation is allowed.

AP Questions

(FR) **2010** BC *3*
(FR) **2011** BC *1*
(MC) **2012** BC *4*
(FR) **2012** BC *2*
(FR) **2014** BC *5c*

What should students be able to do?

- Given the position find the velocity by differentiating; given the velocity find the acceleration by differentiating. This may be in general or at a particular time.
- Given the velocity or acceleration and an initial condition find the velocity or position. This is a type of IVP differential equation question.
- Find the slope of the path
- Determine when the particle is moving left or right ($y'(t) = 0$).
- Determine when the particle is moving up or down ($x'(t) = 0$).
- Find the extreme position (farthest left, right, up or down).
- Find the speed at time t.
- Use the definite integral for arc length to find the distance traveled.
- Find the length of the curve or path.

Shorter questions on these ideas appear in the multiple-choice sections. As always, look over as many questions of this kind from past exams as you can find.

These topics are discussed mainly in chapter 19 of this book.

Type 9: Polar Equations (a BC topic)

Every few years a question about polar equations appears as a free-response on the exam. The question concerns calculus concepts, so knowing the names or graphs of the various curves is not necessary. The graphs are usually given. If the topic is not on the free-response then 1 or maybe 2 questions, probably finding area, can be expected on the multiple-choice section.

AP Questions

(FR) **2005** BC *2*
(FR) **2007** BC *3*
(MC) **2012** BC *91*
(FR) **2013** BC *2*

What should students be able to do?

- Find the intersection of two graphs (to use as limits of integration).
- Find the area enclosed by a graph or between two graphs using the formula.
- Use the formulas $x(\theta) = r(\theta)\cos(\theta)$ and $y(\theta) = r(\theta)\sin(\theta)$ to

  - Convert from polar to rectangular form,
  - Calculate the coordinates of a point on the graph, and
  - Calculate $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ (using the product rule).
- Discuss the motion of a particle moving on the graph by discussing the meaning of $\frac{dr}{d\theta}$ (motion towards or away from the pole), $\frac{dy}{d\theta}$ (motion in the vertical direction) or $\frac{dx}{d\theta}$ (motion in the horizontal direction).
- Find the slope $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ at a point on the graph.

These topics are discussed in chapter 19 of this book.

Type 10: Sequences and Series (a BC topic)

Convergence tests for sequences appear on both sections of the BC Calculus exam. In the multiple-choice section students may be asked to say if a series converges or which of several series converge. On the free-response section there is usually one full question devoted to power series. The Ratio test is used most often to determine the radius of convergence and the other tests to determine if the series convergences at the end points of the interval of convergence.

Students should be familiar with and able to write several terms and the general term of a power series. They may do this by finding the derivatives and constructing the coefficients from them, or they may produce the series by manipulating a known or given series. They may do this by substituting into a series, differentiating it or integrating it.

What should students be able to do?

- Use convergence tests to determine if a series converges. The test to be used is rarely given so students need to know when to use each of the common tests. This may be part the endpoint analysis for the interval of convergence.
- Write the terms of a Taylor or Maclaurin series by calculating the derivatives and constructing the coefficients of each term.
- Distinguish between the Taylor series for a function and the function itself. Do NOT say that the Taylor polynomial is *equal* to the function; say it is *approximately equal*.
- Determine a specific coefficient without writing all the previous coefficients.
- Determine a specific derivative's value from the general form of the coefficient.
- Write a series by substituting into a known series, by differentiating or integrating a known series, or by some other algebraic manipulation of a series.
- Know (memorize) the Maclaurin series for $\sin(x)$, $\cos(x)$, e^x and $\frac{1}{1-x}$ and be able to find other series by substituting into them.

AP Questions

(FR) **2010** BC *6*
(FR) **2011** BC *6*
(FR) **2012** BC *4d, 6*
(MC) **2012** BC *5, 9, 13, 17, 22, 27, 79, 90*
(FR) **2014** BC *6*

- Find the radius and interval of convergence. This is usually done by using the Ratio test and checking the endpoints.
- Be familiar with geometric series, its radius of convergence, and the be able to find the number to which it converges, $S_\infty = \frac{a_1}{1-r}$.
- Re-writing a rational expression as the sum of a geometric series and then writing the series.
- Be familiar with the harmonic series (diverges) and alternating harmonic series (converges).
- Evaluate the first few terms of a series to approximate the value of the function at a point in the interval of convergence.
- Determine the error bound for a convergent series (Alternating series error bound and Lagrange error bound).
- Use the coefficients (the derivatives) to determine information about the function (e.g. extreme values).

This list is long, but only a few of these items can be asked in any given year. The series question on the exam is usually quite straightforward. As I have suggested before, look at and work as many past exam questions to get an idea of what is asked and the difficulty of the questions.

These topics are discussed mainly in chapters 17 and 18 of this book.

AP Question

(FR) **2009** AB *3*, BC *3*

These are the main type of questions asked. There are questions that do not fit into one of these types or fit into more than one. As always, look at past exams.

The final chapter of this book, chapter 23, is for your students. You may photocopy it for them, if you like. It contains hints and suggestions to help them get ready for the exams. Much of this was mentioned previously for you in this book; it is given here, in one place, for you and your students.