A Note on Solving Inequalities

CALCULUS PREREQUISITES: none

COMMON MISUNDERSTANDINGS: This is not so much a misunderstanding as students getting stuck in an inefficient method. The general procedure for solving an inequality involving an expression with several factors is to first find where the expression equals zero or is undefined (where each factor equals zero or is undefined). Then graph these points on a number line. After that students are told to pick a number from each part of the number line and substitute the number into the original expression and determine whether it is positive or negative there. After determining that for each section of the number line one picks the positive or negative ones depending on the inequality.

There is nothing wrong with this method. However, students spend too much time doing the computations when all they need is the sign of the result.

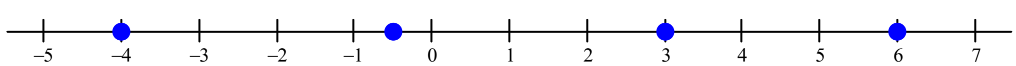
Solving inequalities is done very often in the calculus especially when finding where a function is increasing or decreasing and where it is concave up or down. A quick efficient method is desirable.

Here is an alternative approach.

Example 1 For the first example we will solve

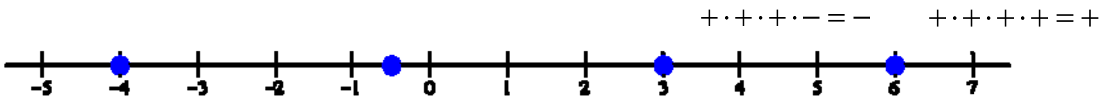
$$(x+4)(2x+1)(x-3)(x-6)>0$$

As before we find where the expression is zero and plot these points on the number line:

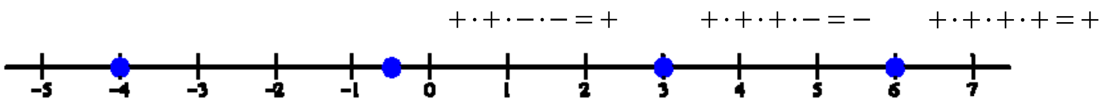


Next consider the right end of the number line. For any value of *x* larger than 6 (like 1,000,000) all the factors are positive, so their product is positive, and so the expression is positive for numbers larger than 6. Mark this with a "+" sign.

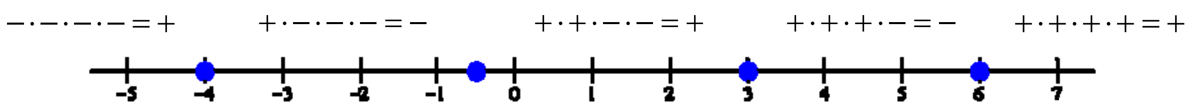
As we move left past *x* = 6 exactly one factor becomes negative, so the product changes sign. The next section is negative.



Moving left past the next zero one more sign changes:



And proceeding to the left end we have:



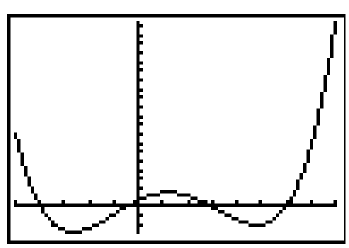
We were asked to find where the expression is positive; we now read the solution from the number line.

$$(x+4)(2x+1)(x-3)(x-6)>0$$

When $x<-4$ or $-\frac{1}{2}<x<3$ or $x>6$

In interval notation: $(-\infty,-4)\cup(-\frac{1}{2},3)\cup(6,\infty)$

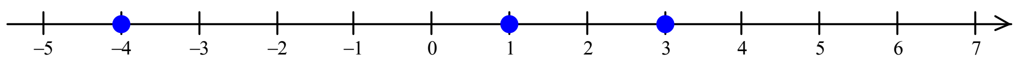
Of course if a calculator is allowed these become even easier: x: [-5, 8] ZoomFit. It is easy to see that the expression is positive when the graph is above the *x*-axis.



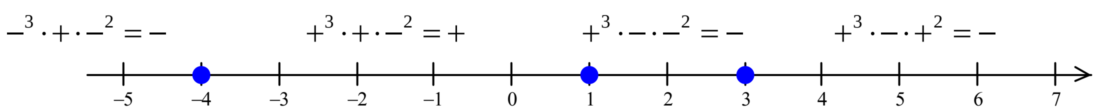
Expressions with repeated factors are handled the same way except that more than one factor changes sign at each zero and the expression sometimes does not change sign. Another fine point is how to handle a factor with a minus sign that looks "backwards" like $(1-x)$. This factor is negative to the right of *x* = 1 and positive to the left. Both are illustrated in the next example.

Example 2: Solve: $(x+4)^3(1-x)(x-3)^2 \le 0$

The number line:

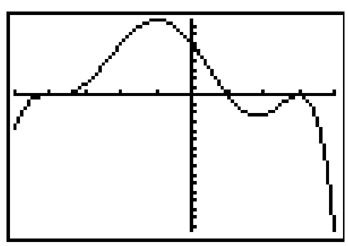


The factor $(1-x)$ is negative on the right end, so the product is negative there also. Moving left past *x* = 3 two signs change and the product remains negative. At *x* = 1 one sign changes (to positive); the expression changes to positive. Finally at *x* = –4 three signs change and the expression becomes negative.



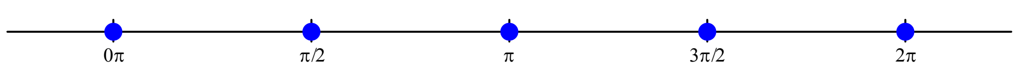
Therefore, $(x+4)^3(1-x)(x-3)^2 \le 0$ has the solution $x \le -4 \text{ or } x \ge 1$

Solution by graphing gives the same result x: [–5, 4] ZoomFit

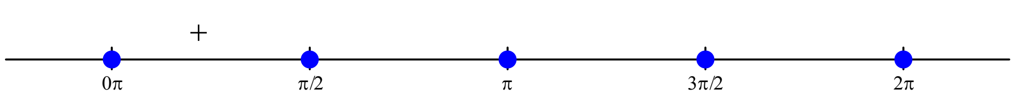


Example 3: Find the values of *x* in the interval $[0, 2\pi]$ for which $\sin(x)\cos(x) \ge 0$

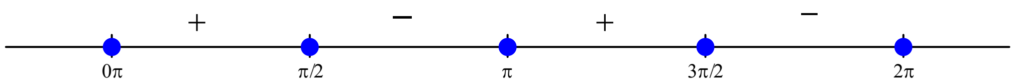
Solution: The zeros are $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



A little to the right of zero both factors are positive:

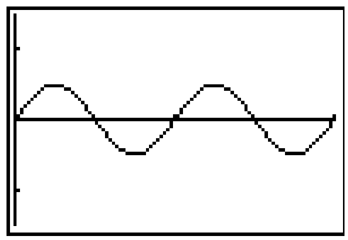


After this, moving right, one sign changes at each zero so:



The solution is $0 \le x \le \frac{\pi}{2}$ or $\pi \le x \le \frac{3\pi}{2}$

The graphical solution is x: $[0, 2\pi]$, y: [–1.5, 1.5]



Other examples: Also see these multiple-choice questions

2008 AB 20 2003 AB 25 1998 AB 19, 22, 24; BC 1, 1997 AB 5, 13; BC 3

An example from the 2007 Practice Exam[1] AB 22

22. If $f'(x) = (x-2)(x-3)^2(x-4)^3$, then *f* has which of the following relative extrema?

I. A relative maximum at $x = 2$
II. A relative minimum at $x = 3$
III. A relative maximum at $x = 4$

(A) I only (B) III only (C) I and III only
(D) II and III only (E) I, II, and III

Answer: (A) The expression does not change sign at $x = 3$.

[1] Please remember that the 2007 "Practice Exam" should be used as a review. Do not pull questions from it for examples, or use parts of it. Do your best to keep it out of student hands.